

YEAR 2 PURE · CHAPTER 17

Vectors in Three Dimensions

Specification: Edexcel Pure 10.1–10.5 — Vectors · AQA section J1–J5

KEY FACTS — THIS CHAPTER

- In 3D, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the x -, y - and z -axes: $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
- The magnitude is $|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$, and a unit vector in the same direction is $\mathbf{a} \div |\mathbf{a}|$.
- The distance between two points is the magnitude of the vector between them: $|\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}|$.
- The angle θ_x that $\mathbf{a}$ makes with the x -axis has $\cos \theta_x = \frac{x}{|\mathbf{a}|}$, and similarly for the other axes.
- Adding, subtracting and scaling work component by component, exactly as in 2D.

Everything from the Year 1 vectors chapter works in three dimensions: a third component joins the other two, and every rule gains a third term. A 3D vector describes a position or a movement in space, such as a drone's flight path or the corner of a room.

1. Coordinates and vectors in 3D

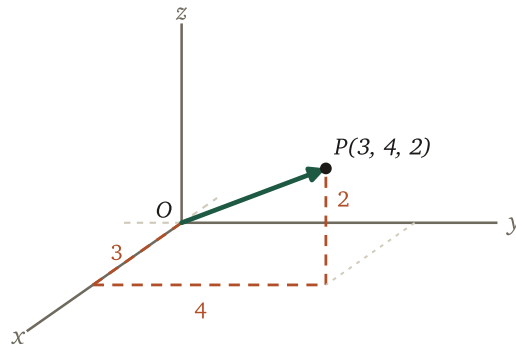
Three axes at right angles meet at the origin O . They are arranged so that if the x -axis points towards you and the y -axis to the right, the z -axis points up.

KEY RESULT

A point P has coordinates (x, y, z) . Its **position vector** is

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the x -, y - and z -axes.



$P(3, 4, 2)$: go 3 along x , 4 along y and 2 up. The arrow OP is $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

2. Magnitude and distance

KEY RESULT

$$|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$$

The distance between A and B is $|\overrightarrow{AB}|$, where $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

PYTHAGORAS TWICE

In the figure above, the point $(3, 4, 0)$ is $\sqrt{3^2 + 4^2} = 5$ from O across the flat xy -plane. P is then 2 straight up from it, at right angles to that line, so $OP = \sqrt{5^2 + 2^2} = \sqrt{3^2 + 4^2 + 2^2}$. The same two steps work for any point.

WORKED EXAMPLE 17.1 · MAGNITUDE AND UNIT VECTOR

For $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, find $|\mathbf{a}|$ and a unit vector in the direction of $\mathbf{a}$.

$$|\mathbf{a}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

Dividing by 7 gives the unit vector $\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$.

WORKED EXAMPLE 17.2 · DISTANCE BETWEEN TWO POINTS

Find the distance between $A(1, -2, 4)$ and $B(3, 1, -2)$.

$$\begin{aligned}\vec{AB} &= \begin{pmatrix} 3 - 1 \\ 1 - (-2) \\ -2 - 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} \\ |\vec{AB}| &= \sqrt{4 + 9 + 36} = 7\end{aligned}$$

3. The angle with each axis

The component along an axis is the magnitude times the cosine of the angle the vector makes with that axis, just as in 2D.

KEY RESULT

If $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes angles θ_x , θ_y and θ_z with the positive x -, y - and z -axes, then

$$\begin{aligned}\cos \theta_x &= \frac{x}{|\mathbf{a}|} \\ \cos \theta_y &= \frac{y}{|\mathbf{a}|} \\ \cos \theta_z &= \frac{z}{|\mathbf{a}|}\end{aligned}$$

WORKED EXAMPLE 17.3 · ANGLES WITH THE AXES

Find the angles that $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ makes with the positive x -, y - and z -axes.

From Example 17.1, $|\mathbf{a}| = 7$, so

$$\begin{aligned}\theta_x &= \cos^{-1} \frac{2}{7} = 73.4^\circ \\ \theta_y &= \cos^{-1} \left(-\frac{3}{7}\right) = 115.4^\circ \\ \theta_z &= \cos^{-1} \frac{6}{7} = 31.0^\circ\end{aligned}$$

(all to 1 d.p.). The negative y -component gives an obtuse angle with the y -axis: the vector points away from it.

4. Vector arithmetic in 3D

KEY RESULT

Add, subtract and multiply by a scalar one component at a time:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\lambda \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}$$

Two vectors are **parallel** when one is a scalar multiple of the other.

WORKED EXAMPLE 17.4 · COMBINING VECTORS

Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{k}$, find $2\mathbf{a} - 3\mathbf{b}$ and its magnitude.

Note that $\mathbf{b}$ has no $\mathbf{j}$ component: it is $0\mathbf{j}$.

$$2\mathbf{a} - 3\mathbf{b} = \begin{pmatrix} 2 - 9 \\ 4 - 0 \\ -2 - 6 \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \\ -8 \end{pmatrix}$$

$$|2\mathbf{a} - 3\mathbf{b}| = \sqrt{49 + 16 + 64} = \sqrt{129}$$

5. Position vectors and geometric problems

The methods from Year 1 carry straight over: write each side as a vector, then use equal vectors for parallelograms, scalar multiples for parallel lines and collinear points, and magnitudes for lengths.

WORKED EXAMPLE 17.5 · COMPLETING A PARALLELOGRAM

$A(1, 2, 3)$, $B(4, 0, 5)$ and $C(2, 1, -1)$ are three corners of the parallelogram $ABCD$. Find the coordinates of D .

1. In $ABCD$, $\overrightarrow{DC} = \overrightarrow{AB}$, so $\mathbf{d} = \mathbf{c} - \overrightarrow{AB}$.

2. $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, so

$$\mathbf{d} = \begin{pmatrix} 2 - 3 \\ 1 + 2 \\ -1 - 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$$

and D is $(-1, 3, -3)$.

WORKED EXAMPLE 17.6 · COLLINEAR POINTS

Show that $P(1, 2, -1)$, $Q(3, 5, 1)$ and $R(7, 11, 5)$ lie on a straight line, and find the ratio $PQ : QR$.

1. $\overrightarrow{PQ} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{QR} = 4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k} = 2\overrightarrow{PQ}$.

2. So $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel, and they share the point Q , so P , Q and R are collinear.

3. QR is twice as long as PQ , so $PQ : QR = 1 : 2$.

COMMON MISTAKE

Parallel is not enough to prove collinear: two parallel vectors could lie on parallel lines that never meet. Always say that they share a point.

WORKED EXAMPLE 17.7 · AN UNKNOWN COMPONENT

The vector $2\mathbf{i} + t\mathbf{j} + 4\mathbf{k}$ has magnitude 6. Find the possible values of t .

$$4 + t^2 + 16 = 36$$

$$t^2 = 16$$

$$t = \pm 4$$

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[2 marks]

Find $|3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}|$.

SOLUTION

$$\sqrt{9 + 16 + 144} = \sqrt{169} = 13.$$

Question 2

[2 marks]

Find a unit vector in the direction of $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

SOLUTION

The magnitude is $\sqrt{1 + 4 + 4} = 3$, so the unit vector is $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

Question 3

[2 marks]

Find the distance between the points $(2, -1, 3)$ and $(-1, 3, 3)$.

SOLUTION

The vector between them is $-3\mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$, so the distance is $\sqrt{9 + 16 + 0} = 5$.

Question 4

[2 marks]

Given $\mathbf{p} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{q} = -\mathbf{i} + 3\mathbf{j}$, find $3\mathbf{p} - 2\mathbf{q}$.

SOLUTION

$$3\mathbf{p} - 2\mathbf{q} = \begin{pmatrix} 6 + 2 \\ -3 - 6 \\ 12 - 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \\ 12 \end{pmatrix}$$

Standard**Question 5**

[4 marks]

Find the angles that $4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ makes with each of the positive coordinate axes.

SOLUTION

The magnitude is $\sqrt{16 + 4 + 16} = 6$, so

$$\theta_x = \cos^{-1} \frac{4}{6} = 48.2^\circ$$

$$\theta_y = \cos^{-1} \left(-\frac{2}{6} \right) = 109.5^\circ$$

$$\theta_z = \cos^{-1} \frac{4}{6} = 48.2^\circ$$

(all to 1 d.p.).

Question 6

[3 marks]

A is $(2, 0, -1)$ and B is $(5, 4, 11)$. Find $\overrightarrow{AB}$, the distance AB and the midpoint of AB .

SOLUTION

$\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$, so $AB = \sqrt{9 + 16 + 144} = 13$. The midpoint is $\left(\frac{2+5}{2}, \frac{0+4}{2}, \frac{-1+11}{2} \right) = (3.5, 2, 5)$.

Question 7

[3 marks]

The vector $t\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$ has magnitude 7. Find the possible values of t .

SOLUTION

$$t^2 + 4t^2 + 4 = 49$$

$$5t^2 = 45$$

So $t^2 = 9$ and $t = \pm 3$.

Question 8

[5 marks]

A is $(1, 1, 1)$, B is $(3, 2, 3)$ and C is $(2, 3, -1)$. Show that triangle ABC is right-angled and isosceles.

SOLUTION

$$\overrightarrow{AB} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad AB = \sqrt{9} = 3$$

$$\overrightarrow{AC} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \quad AC = \sqrt{9} = 3$$

$$\overrightarrow{BC} = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}, \quad BC = \sqrt{18}$$

$AB = AC$, so the triangle is isosceles. $AB^2 + AC^2 = 9 + 9 = 18 = BC^2$, so by the converse of Pythagoras the angle at A is a right angle.

Question 9

[4 marks]

Show that $A(2, -1, 3)$, $B(4, 3, 1)$ and $C(7, 9, -2)$ are collinear, and find the ratio $AB : BC$.

SOLUTION

$\overrightarrow{AB} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{BC} = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} = \frac{3}{2}\overrightarrow{AB}$. They are parallel and share the point B , so A , B and C are collinear, and $AB : BC = 2 : 3$.

Exam style**Question 10**

[6 marks]

$A(1, 2, 0)$, $B(4, 3, 2)$ and $C(6, 1, 5)$ are three corners of the parallelogram $ABCD$.

- (a) Find the coordinates of D . [3]
- (b) Find the exact perimeter of the parallelogram. [3]

SOLUTION

(a) $\overrightarrow{AB} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and $\overrightarrow{DC} = \overrightarrow{AB}$, so $\mathbf{d} = \mathbf{c} - \overrightarrow{AB} = 3\mathbf{i} + 0\mathbf{j} + 3\mathbf{k}$. D is $(3, 0, 3)$.

(b) $AB = \sqrt{9 + 1 + 4} = \sqrt{14}$ and $\overrightarrow{BC} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, so $BC = \sqrt{4 + 4 + 9} = \sqrt{17}$. Opposite sides are equal, so the perimeter is $2\sqrt{14} + 2\sqrt{17}$.

Question 11

[6 marks]

A is $(3, -1, 2)$ and B is $(7, 7, -6)$. The point P lies on AB with $AP : PB = 1 : 3$.

- (a) Find the length of AB . [2]
- (b) Find the coordinates of P . [3]
- (c) Find the distance of P from the origin. [1]

SOLUTION

(a) $\overrightarrow{AB} = 4\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$, so $AB = \sqrt{16 + 64 + 64} = 12$.

(b) P is a quarter of the way from A to B :

$$\mathbf{p} = \mathbf{a} + \frac{1}{4}\overrightarrow{AB} = \begin{pmatrix} 3 + 1 \\ -1 + 2 \\ 2 - 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

so P is $(4, 1, 0)$.

(c) $OP = \sqrt{16 + 1 + 0} = \sqrt{17}$.

Question 12

[7 marks]

A drone takes off from a point O on flat ground and flies in a straight line to the point A , with $\overrightarrow{OA} = 200\mathbf{i} + 150\mathbf{j} + 60\mathbf{k}$ metres. Here $\mathbf{k}$ points vertically up.

- (a) Find the distance OA , to the nearest metre. [2]
- (b) Find the angle between $\overrightarrow{OA}$ and the vertical, and hence the angle at which the drone climbs above the horizontal. [3]
- (c) The drone then flies at a constant height to B , where $\overrightarrow{OB} = 50\mathbf{i} + 350\mathbf{j} + 60\mathbf{k}$. Find the distance AB . [2]

SOLUTION

(a) $OA = \sqrt{200^2 + 150^2 + 60^2} = \sqrt{66\,100} = 257$ m.

(b) $\cos \theta_z = \frac{60}{257.09\dots}$, so $\theta_z = 76.5^\circ$ (1 d.p.) from the vertical, and the drone climbs at $90^\circ - 76.5^\circ = 13.5^\circ$ above the horizontal.

(c) $\overrightarrow{AB} = -150\mathbf{i} + 200\mathbf{j} + 0\mathbf{k}$, so $AB = \sqrt{22\,500 + 40\,000} = 250$ m.