

YEAR 2 PURE • CHAPTER 9

Reciprocal and Inverse Trigonometric Functions

Specification: Edexcel Pure 5.4, 5.5 — Trigonometry • AQA section E4, E5

KEY FACTS — THIS CHAPTER

- $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$ and $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$. The third letter tells you which function is underneath.
- Each reciprocal has an asymptote wherever the function underneath is 0, and is ± 1 wherever it is ± 1 .
- $1 + \tan^2 \theta \equiv \sec^2 \theta$ and $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$. Both come from dividing $\sin^2 \theta + \cos^2 \theta \equiv 1$.
- $\arcsin$, $\arccos$ and $\arctan$ undo $\sin$, $\cos$ and $\tan$ on a restricted domain, so each gives one answer.
- Ranges: $\arcsin x$ and $\arctan x$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$; $\arccos x$ lies between 0 and π .

Three new functions, and three old ones run backwards. The reciprocal functions tidy up expressions and give two more identities for solving equations. The inverse functions are the buttons on your calculator, and this chapter explains why each gives the answer it does.

1. sec, cosec and cot

KEY RESULT

$$\begin{aligned}\sec x &= \frac{1}{\cos x} \\ \operatorname{cosec} x &= \frac{1}{\sin x} \\ \cot x &= \frac{1}{\tan x}\end{aligned}$$

Each is undefined where the function underneath is 0. Since $\tan x = \frac{\sin x}{\cos x}$, you can also write $\cot x = \frac{\cos x}{\sin x}$, which works even where $\tan x$ is undefined.

To remember which is which, look at the third letter: sec goes with cos, cosec with sin, and cot with tan.

WORKED EXAMPLE 9.1 • EXACT VALUES

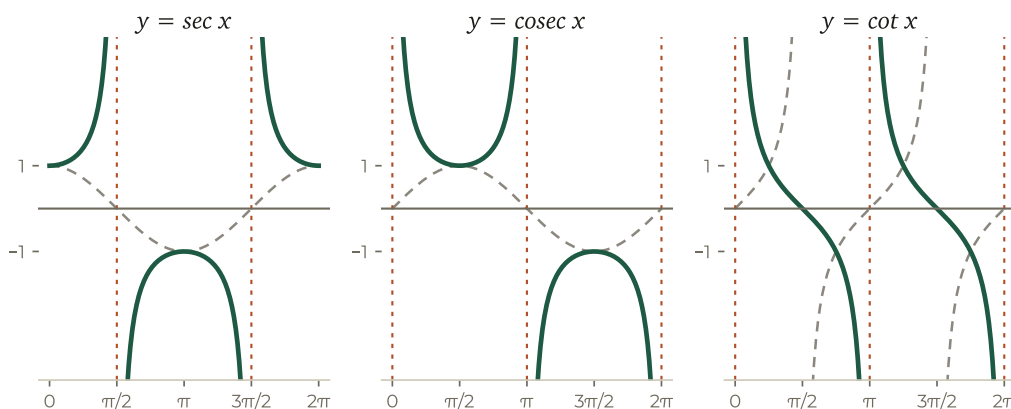
Find the exact values of $\sec \frac{\pi}{6}$, $\operatorname{cosec} \frac{3\pi}{4}$ and $\cot \frac{5\pi}{3}$.

Find the function underneath first, then turn it upside down:

$$\begin{aligned}\sec \frac{\pi}{6} &= \frac{1}{\cos \frac{\pi}{6}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\ \operatorname{cosec} \frac{3\pi}{4} &= \frac{1}{\sin \frac{3\pi}{4}} = \frac{2}{\sqrt{2}} = \sqrt{2} \\ \cot \frac{5\pi}{3} &= \frac{1}{\tan \frac{5\pi}{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}\end{aligned}$$

2. Their graphs

Turning each height upside down gives the shape of the reciprocal graph. Where $\cos x = 1$, $\sec x = 1$ too. As $\cos x$ shrinks towards 0, $\sec x$ grows without limit, and where $\cos x = 0$ there is an asymptote. A reciprocal always has the same sign as the function underneath.



Each reciprocal (solid) over the function underneath it (dashed). Asymptotes sit where the dashed curve crosses the axis.

THE SHAPES

- $y = \sec x$ has period 2π and asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$. Its values are $y \leq -1$ or $y \geq 1$.
- $y = \operatorname{cosec} x$ has period 2π and asymptotes at $x = 0, \pi, 2\pi, \dots$. Its values are $y \leq -1$ or $y \geq 1$.
- $y = \cot x$ has period π , asymptotes at $x = 0, \pi, 2\pi, \dots$ and crosses the x -axis at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

To solve an equation in sec, cosec or cot, turn it into one in cos, sin or tan.

WORKED EXAMPLE 9.2 · AN EQUATION IN COSEC

Solve $\operatorname{cosec} x = -2$ for $0 \leq x \leq 2\pi$.

1. $\operatorname{cosec} x = -2$ means $\sin x = -\frac{1}{2}$.
2. The related acute angle is $\frac{\pi}{6}$. Sine is negative in the third and fourth quadrants, so the solutions are

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

WORKED EXAMPLE 9.3 · A MULTIPLE ANGLE

Solve $\sec 2x = 3$ for $0 \leq x \leq \pi$, giving your answers to 3 significant figures.

1. $\cos 2x = \frac{1}{3}$, and as x runs from 0 to π , $2x$ runs from 0 to 2π .
2. The principal value is $2x = \cos^{-1} \frac{1}{3} = 1.2309\dots$, and the other value in range is $2\pi - 1.2309\dots = 5.0522\dots$
3. Halve each one: $x = 0.615$ or $x = 2.53$ (3 s.f.).

3. Two more identities

Divide every term of $\sin^2 \theta + \cos^2 \theta \equiv 1$ by $\cos^2 \theta$, or by $\sin^2 \theta$:

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 \equiv \frac{1}{\cos^2 \theta}$$

$$1 + \frac{\cos^2 \theta}{\sin^2 \theta} \equiv \frac{1}{\sin^2 \theta}$$

KEY RESULT

$$1 + \tan^2 \theta \equiv \sec^2 \theta$$

$$1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$$

WORKED EXAMPLE 9.4 • FINDING ONE RATIO FROM ANOTHER

Given that $\tan \theta = \frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact value of $\sec \theta$.

1. $\sec^2 \theta = 1 + \frac{9}{16} = \frac{25}{16}$, so $\sec \theta = \pm \frac{5}{4}$.
2. θ is in the third quadrant, where $\cos \theta$ is negative, and $\sec \theta$ has the same sign. So $\sec \theta = -\frac{5}{4}$.

WORKED EXAMPLE 9.5 • A QUADRATIC IN SEC

Solve $2 \tan^2 x + \sec x = 1$ for $0 \leq x < 2\pi$, giving your answers to 3 significant figures where they aren't exact.

1. Replace $\tan^2 x$ with $\sec^2 x - 1$, so that only $\sec x$ is left:

$$2(\sec^2 x - 1) + \sec x = 1$$

$$2 \sec^2 x + \sec x - 3 = 0$$

$$(2 \sec x + 3)(\sec x - 1) = 0$$

2. $\sec x = 1$ means $\cos x = 1$, so $x = 0$.
3. $\sec x = -\frac{3}{2}$ means $\cos x = -\frac{2}{3}$. The principal value is $x = 2.3005\dots$, and the other is $2\pi - 2.3005\dots = 3.9826\dots$. So $x = 0, 2.30$ or 3.98 (3 s.f.).

WORKED EXAMPLE 9.6 • PROVING AN IDENTITY

Prove that $\sec \theta - \cos \theta \equiv \sin \theta \tan \theta$.

Start from the left-hand side and write everything in sin and cos:

$$\begin{aligned}
 \sec \theta - \cos \theta &\equiv \frac{1}{\cos \theta} - \cos \theta \\
 &\equiv \frac{1 - \cos^2 \theta}{\cos \theta} \\
 &\equiv \frac{\sin^2 \theta}{\cos \theta} \\
 &\equiv \sin \theta \times \frac{\sin \theta}{\cos \theta} \\
 &\equiv \sin \theta \tan \theta
 \end{aligned}$$

COMMON MISTAKE

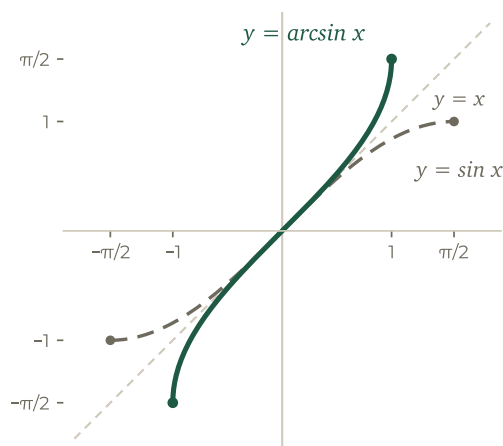
$\sec x$ is not $\cos^{-1} x$. The $^{-1}$ in $\cos^{-1}$ means the inverse function, which is arccos, not the reciprocal. So $\sec 0 = 1$, but $\cos^{-1} 0 = \frac{\pi}{2}$.

4. The inverse functions

$\sin x = \frac{1}{2}$ has infinitely many solutions, so $\sin$ can't be undone as it stands: an inverse function needs a one-to-one function (Chapter 3). The fix is to restrict the domain to one piece of the graph that takes every value once. The inverse of that piece is arcsin, written $\sin^{-1}$ on your calculator.

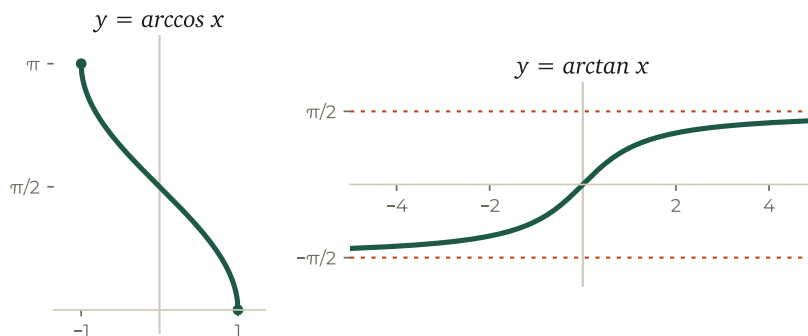
KEY RESULT

	domain	range
$\arcsin x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\arccos x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$\arctan x$	all x	$-\frac{\pi}{2} < y < \frac{\pi}{2}$



$y = \arcsin x$ is the reflection of $\sin x$, restricted to $-\pi/2 \leq x \leq \pi/2$, in the line $y = x$.

As with any inverse function, each graph is the reflection of the restricted function in the line $y = x$. $\arccos$ uses the piece of $\cos x$ from 0 to π , and $\arctan$ the piece of $\tan x$ between its asymptotes at $\pm \frac{\pi}{2}$, which become horizontal asymptotes of $y = \arctan x$.



$y = \arccos x$ (left) and $y = \arctan x$ (right).

WORKED EXAMPLE 9.7 • EXACT INVERSE VALUES

Find the exact values of $\arcsin\left(-\frac{1}{2}\right)$, $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ and $\arctan \sqrt{3}$.

Each answer must lie in the function's range:

$$\begin{aligned}\arcsin\left(-\frac{1}{2}\right) &= -\frac{\pi}{6} \\ \arccos\left(-\frac{\sqrt{2}}{2}\right) &= \frac{3\pi}{4} \\ \arctan \sqrt{3} &= \frac{\pi}{3}\end{aligned}$$

$\arcsin$ gives a negative angle for a negative input. $\arccos$ can't, because its range is 0 to π , so the answer is in the second quadrant instead.

WORKED EXAMPLE 9.8 · A FUNCTION OF AN INVERSE

Given that $y = \arcsin x$, show that $\cos y = \sqrt{1 - x^2}$.

1. $y = \arcsin x$ means $\sin y = x$, with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
2. $\cos^2 y = 1 - \sin^2 y = 1 - x^2$, so $\cos y = \pm\sqrt{1 - x^2}$.
3. Cosine is never negative for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, so take the positive root: $\cos y = \sqrt{1 - x^2}$.

COMMON MISTAKE

$\arcsin(\sin x)$ is only x when x is in the range of $\arcsin$. For example, $\arcsin(\sin \frac{5\pi}{6}) = \arcsin \frac{1}{2} = \frac{\pi}{6}$, not $\frac{5\pi}{6}$.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

Find the exact values of $\sec \frac{\pi}{4}$, $\operatorname{cosec} \frac{5\pi}{6}$ and $\cot \frac{2\pi}{3}$.

SOLUTION

$$\begin{aligned}\sec \frac{\pi}{4} &= \frac{1}{\cos \frac{\pi}{4}} = \frac{2}{\sqrt{2}} = \sqrt{2} \\ \operatorname{cosec} \frac{5\pi}{6} &= \frac{1}{\sin \frac{5\pi}{6}} = \frac{1}{\frac{1}{2}} = 2 \\ \cot \frac{2\pi}{3} &= \frac{1}{\tan \frac{2\pi}{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}\end{aligned}$$

Question 2

[3 marks]

Find the exact values of $\arcsin \frac{\sqrt{3}}{2}$, $\arccos \left(-\frac{1}{2}\right)$ and $\arctan(-1)$.

SOLUTION

Each answer must lie in the function's range:

$$\begin{aligned}\arcsin \frac{\sqrt{3}}{2} &= \frac{\pi}{3} \\ \arccos \left(-\frac{1}{2}\right) &= \frac{2\pi}{3} \\ \arctan(-1) &= -\frac{\pi}{4}\end{aligned}$$

Question 3

[3 marks]

For $0 \leq x \leq 2\pi$, write down the equations of the asymptotes of $y = \operatorname{cosec} x$, and the set of values it takes.

SOLUTION

The asymptotes are where $\sin x = 0$: $x = 0$, $x = \pi$ and $x = 2\pi$.

$\sin x$ takes values from -1 to 1 , so $\operatorname{cosec} x$ takes the values $y \leq -1$ or $y \geq 1$.

Question 4

[2 marks]

Show that $\cot \theta \sec \theta \equiv \operatorname{cosec} \theta$.

SOLUTION

$$\begin{aligned}\cot \theta \sec \theta &\equiv \frac{\cos \theta}{\sin \theta} \times \frac{1}{\cos \theta} \\ &\equiv \frac{1}{\sin \theta} \equiv \operatorname{cosec} \theta\end{aligned}$$

Standard**Question 5**

[3 marks]

Solve $\sec x = -2$ for $0 \leq x \leq 2\pi$.

SOLUTION

$\cos x = -\frac{1}{2}$. The related acute angle is $\frac{\pi}{3}$, and cosine is negative in the second and third quadrants:

$$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}$$

Question 6

[4 marks]

Given that $\sin \theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact values of $\sec \theta$ and $\cot \theta$.

SOLUTION

$\cos^2 \theta = 1 - \frac{25}{169} = \frac{144}{169}$, and $\cos \theta$ is negative in the third quadrant, so $\cos \theta = -\frac{12}{13}$. Then

$$\begin{aligned} \sec \theta &= -\frac{13}{12} \\ \cot \theta &= \frac{\cos \theta}{\sin \theta} = \frac{-12/13}{-5/13} = \frac{12}{5} \end{aligned}$$

Question 7

[3 marks]

Prove that $(1 - \cos^2 \theta)(1 + \cot^2 \theta) \equiv 1$.

SOLUTION

$1 - \cos^2 \theta \equiv \sin^2 \theta$ and $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$, so

$$\begin{aligned} &(1 - \cos^2 \theta)(1 + \cot^2 \theta) \\ &\equiv \sin^2 \theta \times \frac{1}{\sin^2 \theta} \equiv 1 \end{aligned}$$

Question 8

[5 marks]

Solve $\cot^2 x - 3 \operatorname{cosec} x + 3 = 0$ for $0 \leq x \leq 2\pi$, giving exact answers.

SOLUTION

Use $\cot^2 x = \operatorname{cosec}^2 x - 1$, so that only $\operatorname{cosec} x$ is left:

$$\begin{aligned} \operatorname{cosec}^2 x - 3 \operatorname{cosec} x + 2 &= 0 \\ (\operatorname{cosec} x - 1)(\operatorname{cosec} x - 2) &= 0 \end{aligned}$$

$\operatorname{cosec} x = 1$ means $\sin x = 1$, so $x = \frac{\pi}{2}$. $\operatorname{cosec} x = 2$ means $\sin x = \frac{1}{2}$, so $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. So $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

Question 9

[4 marks]

Find the exact values of

(a) $\cos(\arcsin \frac{3}{5})$ [2]

(b) $\tan(\arccos(-\frac{1}{4}))$ [2]

SOLUTION

(a) Let $\alpha = \arcsin \frac{3}{5}$, so $\sin \alpha = \frac{3}{5}$ with $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, where cosine is not negative. So $\cos \alpha = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$.

(b) Let $\beta = \arccos(-\frac{1}{4})$, so $\cos \beta = -\frac{1}{4}$ with $0 \leq \beta \leq \pi$, where sine is not negative. So $\sin \beta = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$ and

$$\tan \beta = \frac{\sqrt{15}/4}{-1/4} = -\sqrt{15}$$

Exam style**Question 10**

[6 marks]

Solve $2 \sec^2 \theta - 3 \tan \theta = 1$ for $0 \leq \theta < 2\pi$. Give exact answers where you can, and the others to 3 significant figures.

SOLUTION

Use $\sec^2 \theta = 1 + \tan^2 \theta$:

$$2(1 + \tan^2 \theta) - 3 \tan \theta = 1$$

$$2 \tan^2 \theta - 3 \tan \theta + 1 = 0$$

$$(2 \tan \theta - 1)(\tan \theta - 1) = 0$$

- $\tan \theta = 1$ gives $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.
- $\tan \theta = \frac{1}{2}$ gives $\theta = 0.4636 \dots$ or $0.4636 \dots + \pi = 3.6052 \dots$

So $\theta = 0.464, \frac{\pi}{4}, 3.61, \frac{5\pi}{4}$.

Question 11

[7 marks]

- (a) Describe the graph of $y = \arccos x$, giving its domain, its range and the coordinates of its end points. [3]
- (b) Solve $\arccos x = \frac{2\pi}{3}$. [1]
- (c) Prove that $\arcsin x + \arccos x = \frac{\pi}{2}$ for $-1 \leq x \leq 1$. [3]

SOLUTION

- (a) The domain is $-1 \leq x \leq 1$ and the range is $0 \leq y \leq \pi$. The graph falls from $(-1, \pi)$ through $(0, \frac{\pi}{2})$ to $(1, 0)$.
- (b) $x = \cos \frac{2\pi}{3} = -\frac{1}{2}$.
- (c) Let $y = \arcsin x$, so $x = \sin y$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Then $\cos(\frac{\pi}{2} - y) = \sin y = x$, and $0 \leq \frac{\pi}{2} - y \leq \pi$, which is the range of $\arccos$. So

$$\arccos x = \frac{\pi}{2} - y = \frac{\pi}{2} - \arcsin x$$

which rearranges to $\arcsin x + \arccos x = \frac{\pi}{2}$.

Question 12

[8 marks]

- (a) Prove that $(\sec \theta + \tan \theta)(\sec \theta - \tan \theta) \equiv 1$. [2]
- (b) Given that $\sec \theta + \tan \theta = 3$, find the exact values of $\sec \theta - \tan \theta$, $\cos \theta$ and $\tan \theta$. [4]
- (c) Hence find θ for $0 \leq \theta < 2\pi$, to 3 significant figures. [2]

SOLUTION

- (a) Multiplying out gives the difference of two squares: $(\sec \theta + \tan \theta)(\sec \theta - \tan \theta) \equiv \sec^2 \theta - \tan^2 \theta \equiv 1$, using $1 + \tan^2 \theta \equiv \sec^2 \theta$.
- (b) From (a), $\sec \theta - \tan \theta = \frac{1}{3}$. Adding and subtracting the two equations:

$$\begin{aligned} 2 \sec \theta &= 3 + \frac{1}{3} = \frac{10}{3} \\ 2 \tan \theta &= 3 - \frac{1}{3} = \frac{8}{3} \end{aligned}$$

So $\sec \theta = \frac{5}{3}$, which gives $\cos \theta = \frac{3}{5}$, and $\tan \theta = \frac{4}{3}$.

- (c) $\cos \theta$ and $\tan \theta$ are both positive, so θ is in the first quadrant only: $\theta = \arccos \frac{3}{5} = 0.927$ (3 s.f.).