

YEAR 2 PURE • CHAPTER 8

Radians

Specification: Edexcel Pure 5.1–5.3 — Trigonometry • AQA section E1, E2, E3

KEY FACTS — THIS CHAPTER

- One **radian** is the angle at the centre of a circle when the arc is as long as the radius. π radians = 180° .
- To convert, multiply degrees by $\frac{\pi}{180}$, or radians by $\frac{180}{\pi}$.
- In radians, sin and cos have period 2π , and tan has period π . Set your calculator to radians.
- For a sector of radius r and angle θ radians: **arc length** $s = r\theta$ and **area** $A = \frac{1}{2}r^2\theta$.
- A **segment** has area $\frac{1}{2}r^2(\theta - \sin \theta)$: the sector minus the triangle.
- For small θ in radians: $\sin \theta \approx \theta$, $\tan \theta \approx \theta$ and $\cos \theta \approx 1 - \frac{1}{2}\theta^2$.

There is nothing special about 360 degrees in a full turn: it's a choice made thousands of years ago. Radians measure an angle by the circle itself, so the formulas for arcs and sectors lose their stray $\frac{\theta}{360}$ factors. From this chapter on, angles are in radians unless a question gives them in degrees.

1. Measuring angles in radians

KEY RESULT

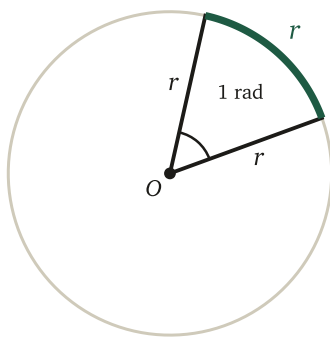
An angle of **one radian** is the angle at the centre of a circle when the arc it cuts off is exactly one radius long.

A full circle has circumference $2\pi r$, which is 2π radii, so a full turn is 2π radians and half a turn is π radians:

$$2\pi \text{ radians} = 360^\circ$$

$$\pi \text{ radians} = 180^\circ$$

One radian is $\frac{180^\circ}{\pi} \approx 57.3^\circ$.



When the arc is as long as the radius, the angle is one radian.

To change degrees into radians, multiply by $\frac{\pi}{180}$. To change radians into degrees, multiply by $\frac{180}{\pi}$. The common angles are simple fractions of π , so learn them both ways.

ANGLES TO KNOW

degrees	30°	45°	60°	90°
radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

The rest follow from these and $180^\circ = \pi$: $120^\circ = \frac{2\pi}{3}$, $135^\circ = \frac{3\pi}{4}$, $150^\circ = \frac{5\pi}{6}$, $270^\circ = \frac{3\pi}{2}$ and $360^\circ = 2\pi$.

WORKED EXAMPLE 8.1 • CONVERTING

Convert 150° to radians, and $\frac{7\pi}{4}$ and 2.5 radians to degrees.

$$150^\circ = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$$

$$\frac{7\pi}{4} = \frac{7\pi}{4} \times \frac{180}{\pi} = 315^\circ$$

$$2.5 = 2.5 \times \frac{180}{\pi} = 143.2^\circ \text{ (1 d.p.)}$$

An angle in radians is often written with no unit at all: an angle of 2.5 means 2.5 radians. Some books write 2.5° or 2.5 rad.

COMMON MISTAKE

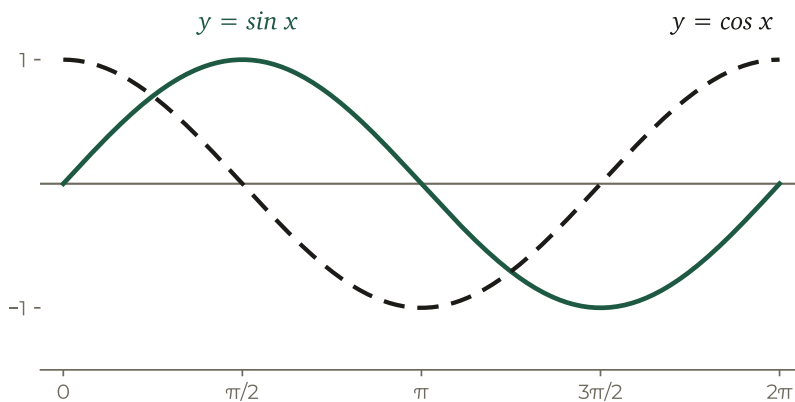
Check your calculator's angle mode before every trig calculation. In radian mode, $\sin 30 = -0.988$, because the calculator reads 30 as 30 radians, which is nearly five full turns.

2. Exact values, graphs and equations in radians

Everything from Year 1 carries over; only the labels change. The exact values are the same numbers at the same angles:

EXACT VALUES IN RADIANS

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	—



sin and cos in radians: one full wave every 2π .

The graphs have the same shapes, with the axis now marked in multiples of π . $\sin x$ and $\cos x$ repeat every 2π . $\tan x$ repeats every π , with asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$. The symmetries you used in degrees become

$$\sin(\pi - x) = \sin x$$

$$\cos(2\pi - x) = \cos x$$

$$\tan(\pi + x) = \tan x$$

WORKED EXAMPLE 8.2 · EXACT VALUES PAST A RIGHT ANGLE

Find the exact values of $\sin \frac{4\pi}{3}$ and $\cos \frac{7\pi}{4}$.

1. $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$ is in the third quadrant, where sine is negative:

$$\sin \frac{4\pi}{3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

2. $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ is in the fourth quadrant, where cosine is positive:

$$\cos \frac{7\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

WORKED EXAMPLE 8.3 · A MULTIPLE ANGLE IN RADIANs

Solve $\sin 2x = \frac{\sqrt{3}}{2}$ for $0 \leq x \leq 2\pi$.

- As x runs from 0 to 2π , $2x$ runs from 0 to 4π . Solve in this wider interval.
- The principal value is $2x = \frac{\pi}{3}$, and sine is also positive at $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Adding 2π to each gives the rest:

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

3. Halve each one:

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

This is Year 1's Example 9.3 in radians: the answers are 30° , 60° , 210° and 240° again.

WORKED EXAMPLE 8.4 · AN ANSWER THAT ISN'T EXACT

Solve $3 \tan x = 5$ for $0 \leq x < 2\pi$, giving your answers to 3 significant figures.

1. $\tan x = \frac{5}{3}$, so the principal value is $x = \tan^{-1} \frac{5}{3} = 1.0303 \dots$

2. $\tan$ repeats every π , so the other solution in range is $1.0303 \dots + \pi = 4.1719 \dots$. So $x = 1.03$ or $x = 4.17$ (3 s.f.).

3. Arcs, sectors and segments

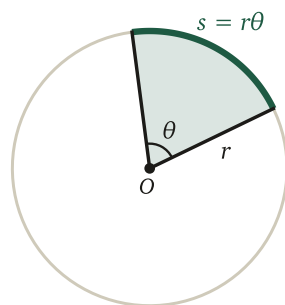
A sector with angle θ is the fraction $\frac{\theta}{2\pi}$ of the whole circle. So its arc is that fraction of the circumference, and its area is that fraction of the circle's area.

KEY RESULT

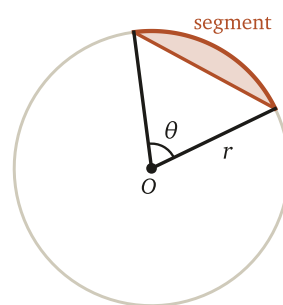
For a sector of radius r and angle θ in radians,

$$s = \frac{\theta}{2\pi} \times 2\pi r = r\theta$$

$$A = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}r^2\theta$$



$$\text{sector: } A = \frac{1}{2}r^2\theta$$



$$\text{segment: } A = \frac{1}{2}r^2(\theta - \sin \theta)$$

A sector (left), and the segment cut off by its chord (right).

WORKED EXAMPLE 8.5 · A SECTOR

A sector has radius 6 cm and angle 1.2 radians. Find its arc length, area and perimeter.

$$s = 6 \times 1.2 = 7.2 \text{ cm}$$

$$A = \frac{1}{2} \times 6^2 \times 1.2 = 21.6 \text{ cm}^2$$

The perimeter is the arc plus two radii: $7.2 + 6 + 6 = 19.2 \text{ cm}$.

WORKED EXAMPLE 8.6 · WORKING BACKWARDS

A sector has area 15 cm^2 and perimeter 16 cm . Find the possible values of its radius r and angle θ .

1. The two facts give two equations:

$$\frac{1}{2}r^2\theta = 15 \quad \text{and} \quad 2r + r\theta = 16$$

2. From the second, $r\theta = 16 - 2r$. Substitute into the first, written as $\frac{1}{2}r(r\theta) = 15$:

$$\begin{aligned}\frac{1}{2}r(16 - 2r) &= 15 \\ r^2 - 8r + 15 &= 0 \\ (r - 3)(r - 5) &= 0\end{aligned}$$

3. If $r = 3$, then $3\theta = 10$ and $\theta = \frac{10}{3}$. If $r = 5$, then $5\theta = 6$ and $\theta = 1.2$. Both angles are less than 2π , so both sectors exist.

A chord cuts a circle into two **segments**. The minor segment is what's left of the sector when you take away the triangle made by the two radii and the chord. The triangle has area $\frac{1}{2}ab \sin C = \frac{1}{2}r^2 \sin \theta$.

KEY RESULT

The segment cut off by a chord that makes angle θ at the centre has area

$$\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = \frac{1}{2}r^2(\theta - \sin \theta)$$

WORKED EXAMPLE 8.7 · A SEGMENT

A chord of a circle of radius 8 cm makes an angle of $\frac{\pi}{3}$ at the centre. Find the exact area of the minor segment, and give it to 3 significant figures.

$$\begin{aligned}A &= \frac{1}{2} \times 8^2 \left(\frac{\pi}{3} - \sin \frac{\pi}{3} \right) \\ &= 32 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \\ &= \frac{32\pi}{3} - 16\sqrt{3} \\ &= 5.80 \text{ cm}^2 \text{ (3 s.f.)}\end{aligned}$$

COMMON MISTAKE

The formulas $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ only work with θ in radians. If a question gives the angle in degrees, convert it first.

4. Small-angle approximations

When θ is small and in radians, $\sin \theta$, $\tan \theta$ and θ itself are almost equal, and $\cos \theta$ is almost 1.

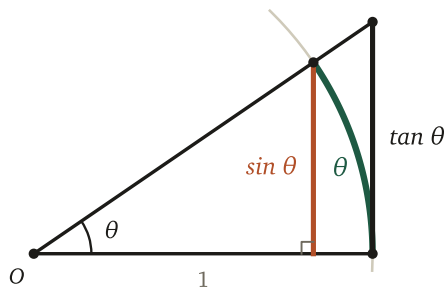
KEY RESULT

For small θ , measured in radians,

$$\sin \theta \approx \theta$$

$$\tan \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{1}{2}\theta^2$$



On a circle of radius 1: $\sin \theta < \theta < \tan \theta$. As θ shrinks, the three lengths close up.

WHERE THE APPROXIMATIONS COME FROM

On a circle of radius 1, the arc has length θ . The height of its end above the first radius is $\sin \theta$, which is a little shorter. The tangent at the start, cut off by the second radius, has length $\tan \theta$, which is a little longer. So $\sin \theta < \theta < \tan \theta$, and as θ shrinks the three lengths close up.

For cosine, use $\cos^2 \theta = 1 - \sin^2 \theta \approx 1 - \theta^2$. Then the binomial series from Chapter 7 gives $\cos \theta \approx (1 - \theta^2)^{\frac{1}{2}} \approx 1 - \frac{1}{2}\theta^2$.

WORKED EXAMPLE 8.8 · SIMPLIFYING AN EXPRESSION

Show that, when θ is small, $\frac{\sin 4\theta + \cos 2\theta - 1}{3 \tan \theta} \approx \frac{4}{3} - \frac{2}{3}\theta$.

1. Replace each function, keeping the whole angle together: $\sin 4\theta \approx 4\theta$, $\tan \theta \approx \theta$ and $\cos 2\theta \approx 1 - \frac{1}{2}(2\theta)^2 = 1 - 2\theta^2$.
2. Substitute and simplify:

$$\begin{aligned} & \frac{\sin 4\theta + \cos 2\theta - 1}{3 \tan \theta} \\ & \approx \frac{4\theta + (1 - 2\theta^2) - 1}{3\theta} \\ & = \frac{4\theta - 2\theta^2}{3\theta} = \frac{4}{3} - \frac{2}{3}\theta \end{aligned}$$

COMMON MISTAKE

Square the whole angle in the cosine approximation: $\cos 2\theta \approx 1 - \frac{1}{2}(2\theta)^2 = 1 - 2\theta^2$, not $1 - \theta^2$.

WORKED EXAMPLE 8.9 · HOW GOOD ARE THEY?

Compare each approximation with the true value when $\theta = 0.1$, and find the percentage errors.

	Value	Estimate	Error
$\sin 0.1$	0.099833	0.1	0.17%
$\tan 0.1$	0.100335	0.1	0.33%
$\cos 0.1$	0.995004	0.995	0.0004%

Each error is the difference as a percentage of the true value, for example $\frac{0.1 - 0.099833}{0.099833} \times 100 = 0.17\%$. All three are good, and the cosine one is far better, because its first missing term is a θ^4 term.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[3 marks]

- (a) Convert 240° to radians, as an exact multiple of π . [1]
- (b) Convert $\frac{7\pi}{12}$ radians to degrees. [1]
- (c) Convert 0.8 radians to degrees, to 1 decimal place. [1]

SOLUTION

- (a) $240 \times \frac{\pi}{180} = \frac{4\pi}{3}$.
- (b) $\frac{7\pi}{12} \times \frac{180}{\pi} = 105^\circ$.
- (c) $0.8 \times \frac{180}{\pi} = 45.8^\circ$ (1 d.p.).

Question 2

[4 marks]

- (a) Find the exact values of $\sin \frac{5\pi}{4}$, $\cos \frac{2\pi}{3}$ and $\tan \frac{5\pi}{3}$. [3]
- (b) State the period of $y = \cos 3x$, in radians. [1]

SOLUTION

- (a) Each is the related acute angle's value, with the sign of its quadrant:

$$\begin{aligned}\sin \frac{5\pi}{4} &= -\sin \frac{\pi}{4} = -\frac{\sqrt{2}}{2} \\ \cos \frac{2\pi}{3} &= -\cos \frac{\pi}{3} = -\frac{1}{2} \\ \tan \frac{5\pi}{3} &= -\tan \frac{\pi}{3} = -\sqrt{3}\end{aligned}$$

- (b) $\cos x$ has period 2π , so $\cos 3x$ has period $\frac{2\pi}{3}$.

Question 3

[3 marks]

A sector has radius 5 cm and angle 0.9 radians. Find its arc length, its area and its perimeter.

SOLUTION

$$s = 5 \times 0.9 = 4.5 \text{ cm}$$

$$A = \frac{1}{2} \times 5^2 \times 0.9 = 11.25 \text{ cm}^2$$

The perimeter is $4.5 + 5 + 5 = 14.5 \text{ cm}$.

Question 4

[3 marks]

Find the approximate value of $\frac{\theta \sin \theta}{1 - \cos \theta}$ when θ is small.

SOLUTION

$\sin \theta \approx \theta$ and $1 - \cos \theta \approx \frac{1}{2}\theta^2$, so

$$\frac{\theta \sin \theta}{1 - \cos \theta} \approx \frac{\theta^2}{\frac{1}{2}\theta^2} = 2$$

Standard**Question 5**

[3 marks]

Solve $\cos x = -0.4$ for $0 \leq x \leq 2\pi$, giving your answers to 3 significant figures.

SOLUTION

The principal value is $x = \cos^{-1}(-0.4) = 1.9823 \dots$. The other solution comes from the symmetry $\cos(2\pi - x) = \cos x$: $2\pi - 1.9823 \dots = 4.3008 \dots$

So $x = 1.98$ or $x = 4.30$ (3 s.f.).

Question 6

[4 marks]

Solve $2 \sin^2 x = \sin x$ for $0 \leq x \leq 2\pi$, giving exact answers.

SOLUTION

Don't divide by $\sin x$: that would lose the solutions where $\sin x = 0$. Factorise instead:

$$\sin x (2 \sin x - 1) = 0$$

- $\sin x = 0$ gives $x = 0, \pi, 2\pi$.
- $\sin x = \frac{1}{2}$ gives $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

So $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$.

Question 7

[3 marks]

A sector of a circle of radius 8 cm has area 24 cm^2 . Find the angle of the sector and its perimeter.

SOLUTION

$\frac{1}{2} \times 8^2 \times \theta = 24$, so $32\theta = 24$ and $\theta = 0.75$ radians.

The arc is $8 \times 0.75 = 6 \text{ cm}$, so the perimeter is $6 + 8 + 8 = 22 \text{ cm}$.

Question 8

[5 marks]

A sector of a circle has radius $r \text{ cm}$ and perimeter 20 cm.

- Show that the area of the sector is $A = 10r - r^2$. [3]
- Find the largest possible area, and the angle of the sector when the area is largest. [2]

SOLUTION

(a) The perimeter is $2r + r\theta = 20$, so $r\theta = 20 - 2r$. Then

$$\begin{aligned} A &= \frac{1}{2}r^2\theta = \frac{1}{2}r(r\theta) \\ &= \frac{1}{2}r(20 - 2r) \\ &= 10r - r^2 \end{aligned}$$

(b) Complete the square: $A = 25 - (r - 5)^2$. So the largest area is 25 cm^2 , when $r = 5$. Then $5\theta = 20 - 10$, so $\theta = 2$ radians.

Question 9

[5 marks]

- Show that, when θ is small, $\frac{\cos 3\theta - 1}{\theta \sin 2\theta} \approx -\frac{9}{4}$. [3]
- Find the value of the expression when $\theta = 0.05$, to 4 decimal places, and the percentage error in the approximation. [2]

SOLUTION

(a) $\cos 3\theta \approx 1 - \frac{1}{2}(3\theta)^2 = 1 - \frac{9}{2}\theta^2$ and $\sin 2\theta \approx 2\theta$, so

$$\frac{\cos 3\theta - 1}{\theta \sin 2\theta} \approx \frac{-\frac{9}{2}\theta^2}{2\theta^2} = -\frac{9}{4}$$

(b) In radian mode, the expression is -2.2495 (4 d.p.) when $\theta = 0.05$. The percentage error is

$$\frac{2.25 - 2.2495\dots}{2.2495\dots} \times 100 = 0.02\%$$

to 1 significant figure.

Exam style**Question 10**

[7 marks]

From a point on level ground 400 m from the foot of a tall building, the angle of elevation of the top of the building is 0.05 radians.

- (a) Use a small-angle approximation to estimate the height of the building. [2]
- (b) Find the height to 5 significant figures, and the percentage error in your estimate. [3]
- (c) A second building, also 400 m away, has an angle of elevation of 0.5 radians. Explain why the same method would be much less reliable for it. [2]

SOLUTION

(a) The height is $h = 400 \tan 0.05$. For small angles $\tan \theta \approx \theta$, so $h \approx 400 \times 0.05 = 20$ m.

(b) $h = 400 \tan 0.05 = 20.017$ m (5 s.f.). The percentage error is

$$\frac{20.017 - 20}{20.017} \times 100 = 0.08\%$$

to 1 significant figure.

(c) The approximation $\tan \theta \approx \theta$ only works when θ is small. At 0.5 radians, $\tan 0.5 = 0.546$, so the estimate $400 \times 0.5 = 200$ m is about 8% below the true height of 219 m.

Question 11

[5 marks]

Solve $2 \cos^2 \theta + 3 \sin \theta = 3$ for $0 \leq \theta < 2\pi$, giving exact answers.

SOLUTION

Use $\cos^2 \theta = 1 - \sin^2 \theta$ to get an equation in $\sin \theta$ only:

$$2(1 - \sin^2 \theta) + 3 \sin \theta = 3$$

$$2 \sin^2 \theta - 3 \sin \theta + 1 = 0$$

$$(2 \sin \theta - 1)(\sin \theta - 1) = 0$$

$\sin \theta = \frac{1}{2}$ gives $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, and $\sin \theta = 1$ gives $\frac{\pi}{2}$. So $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

Question 12

[9 marks]

A sector OAB of a circle with centre O has area 12 cm^2 and arc length 6 cm .

- (a) Find the radius of the circle and the angle AOB . [3]
- (b) Find the area of the segment between the chord AB and the arc, to 3 significant figures. [3]
- (c) Find the perimeter of the segment, to 3 significant figures. [3]

SOLUTION

(a) $r\theta = 6$ and $\frac{1}{2}r^2\theta = 12$. Writing the area as $\frac{1}{2}r(r\theta) = \frac{1}{2}r \times 6 = 3r$ gives $3r = 12$, so $r = 4 \text{ cm}$ and $\theta = \frac{6}{4} = 1.5$ radians.

(b)

$$\begin{aligned} A &= \frac{1}{2} \times 4^2 \times (1.5 - \sin 1.5) \\ &= 8(1.5 - 0.9974\dots) \\ &= 4.02 \text{ cm}^2 \text{ (3 s.f.)} \end{aligned}$$

(c) Find the chord with the cosine rule:

$$\begin{aligned} AB^2 &= 4^2 + 4^2 - 2 \times 4 \times 4 \cos 1.5 \\ &= 29.736\dots \\ AB &= 5.4531\dots \end{aligned}$$

The perimeter is the arc plus the chord: $6 + 5.4531\dots = 11.5 \text{ cm}$ (3 s.f.).