

YEAR 2 PURE · CHAPTER 2

Algebraic Fractions and Partial Fractions

Specification: Edexcel Pure 2.6, 2.10 — Algebra and functions · AQA section B6, B10

KEY FACTS — THIS CHAPTER

- **Simplify** an algebraic fraction by factorising the top and bottom fully, then cancelling common **factors**, never terms.
- **Multiply** by multiplying tops and bottoms (cancel first). **Divide** by multiplying by the reciprocal.
- **Add or subtract** over a common denominator, ideally the lowest common multiple of the denominators.
- **Partial fractions** reverse the adding: $\frac{px+q}{(x-a)(x-b)} \equiv \frac{A}{x-a} + \frac{B}{x-b}$.
- A **repeated** factor needs a term for each power: $\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$.
- Find the constants by **substituting** the roots of the denominator, then **comparing coefficients** for any left over.
- If the top's degree is at least the bottom's, the fraction is **improper**: divide first.

Algebraic fractions follow the same rules as number fractions: you cancel common factors, find common denominators, and flip to divide. The new idea in this chapter runs the other way. Partial fractions take one complicated fraction apart into simple ones, and those simple pieces are what the binomial series (Chapter 7) and integration (Chapter 15) can handle.

1. Simplifying, multiplying and dividing

KEY RESULT

1. **Factorise** the numerator and denominator completely.
2. **Cancel** any factor that appears in both.
3. To **multiply**, multiply the numerators and the denominators, cancelling first. To **divide**, multiply by the reciprocal of the second fraction.

WORKED EXAMPLE 2.1 · CANCELLING A COMMON FACTOR

Simplify $\frac{x^2 - 9}{x^2 + x - 6}$.

1. Factorise the top (a difference of two squares) and the bottom:

$$\frac{x^2 - 9}{x^2 + x - 6} = \frac{(x - 3)(x + 3)}{(x + 3)(x - 2)}$$

2. Cancel the common factor $(x + 3)$:

$$= \frac{x - 3}{x - 2}$$

COMMON MISTAKE

Cancel **factors**, not terms. In $\frac{x + 3}{x + 5}$ nothing cancels: the x s are terms of a sum, not factors of the whole top and bottom. Try $x = 1$: the fraction is $\frac{4}{6}$, not $\frac{3}{5}$.

WORKED EXAMPLE 2.2 · DIVIDING FRACTIONS

Simplify $\frac{x^2 - 4}{3x + 6} \div \frac{x - 2}{6x}$.

1. Factorise, and multiply by the reciprocal of the second fraction:

$$\frac{(x - 2)(x + 2)}{3(x + 2)} \times \frac{6x}{x - 2}$$

2. Cancel $(x - 2)$, $(x + 2)$ and the 3 into the 6:

$$= 2x$$

2. Adding and subtracting

KEY RESULT

Rewrite each fraction over a **common denominator**, then combine the numerators:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

If the denominators already share a factor, use their **lowest common multiple** instead of the product: the working is shorter and the answer needs less cancelling.

WORKED EXAMPLE 2.3 · SUBTRACTING TWO FRACTIONS

Write $\frac{3}{x+2} - \frac{1}{x-1}$ as a single fraction.

1. The common denominator is $(x + 2)(x - 1)$:

$$\frac{3(x-1)}{(x+2)(x-1)} - \frac{x+2}{(x+2)(x-1)}$$

2. Combine the numerators, keeping the bracket on the second one:

$$\begin{aligned} &= \frac{3x - 3 - x - 2}{(x+2)(x-1)} \\ &= \frac{2x - 5}{(x+2)(x-1)} \end{aligned}$$

COMMON MISTAKE

The minus sign applies to the **whole** second numerator. Writing $3x - 3 - x + 2$ instead of $3x - 3 - x - 2$ is the most common slip in these questions. Put the numerator in a bracket before you expand.

3. Partial fractions

Adding $\frac{2}{x-1} + \frac{3}{x+2}$ gives $\frac{5x+1}{(x-1)(x+2)}$. Partial fractions run that backwards: given the single fraction, find the simple ones that add up to it.

PARTIAL FRACTIONS: DISTINCT LINEAR FACTORS

A proper fraction whose denominator is a product of different linear factors splits into one term for each factor:

$$\frac{px+q}{(x-a)(x-b)} \equiv \frac{A}{x-a} + \frac{B}{x-b}$$

With three factors there are three terms, $\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$. To find A and B , multiply through by the denominator and **substitute** $x = a$ and $x = b$ in turn: each value makes all but one term vanish.

WORKED EXAMPLE 2.4 · TWO LINEAR FACTORS

Express $\frac{5x+1}{(x-1)(x+2)}$ in partial fractions.

1. Write down the form:

$$\frac{5x+1}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2}$$

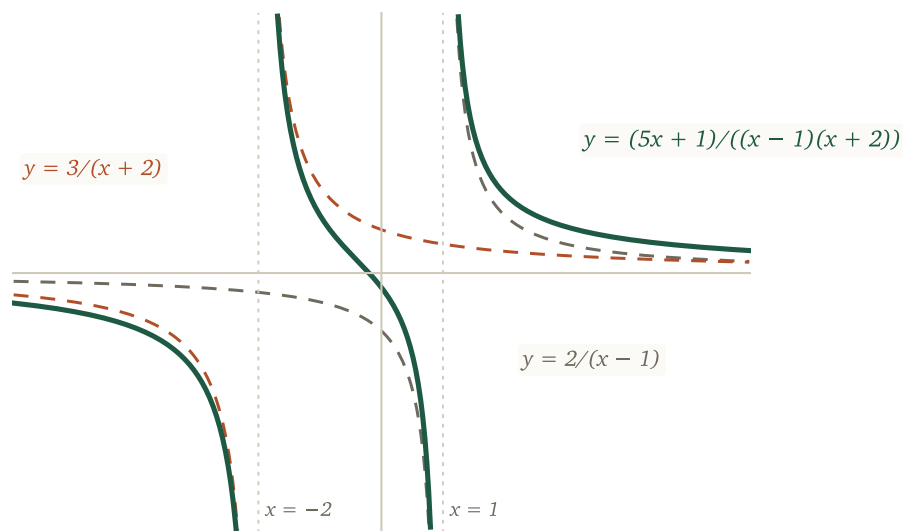
2. Multiply through by $(x-1)(x+2)$:

$$5x+1 \equiv A(x+2) + B(x-1)$$

3. Substitute $x = 1$: $6 = 3A$, so $A = 2$. Substitute $x = -2$: $-9 = -3B$, so $B = 3$.

4. So

$$\frac{5x+1}{(x-1)(x+2)} \equiv \frac{2}{x-1} + \frac{3}{x+2}$$



Near $x = 1$ the curve follows $2/(x - 1)$; near $x = -2$ it follows $3/(x + 2)$.

WHY SUBSTITUTING WORKS

The line $5x + 1 \equiv A(x + 2) + B(x - 1)$ is an **identity**: true for every value of x , so you may choose any value you like. Choosing $x = 1$ wipes out the B term, leaving A on its own. In effect you are **covering up** the factor $(x - 1)$ in the original fraction and putting $x = 1$ into what is left:

$$A = \frac{5(1) + 1}{1 + 2} = 2.$$

4. Repeated factors

A squared factor such as $(x - 1)^2$ needs two terms, one over $(x - 1)$ and one over $(x - 1)^2$. Leaving out either one makes the split impossible.

PARTIAL FRACTIONS: A REPEATED FACTOR

$$\begin{aligned} \frac{px^2 + qx + r}{(x - a)^2(x - b)} \\ \equiv \frac{A}{x - a} + \frac{B}{(x - a)^2} + \frac{C}{x - b} \end{aligned}$$

Substituting $x = a$ and $x = b$ finds B and C . For A , **compare coefficients**, usually of x^2 .

WORKED EXAMPLE 2.5 · A REPEATED FACTOR

Express $\frac{4x^2 - 3x + 5}{(x - 1)^2(x + 2)}$ in partial fractions.

1. Write down the form:

$$\begin{aligned} \frac{4x^2 - 3x + 5}{(x - 1)^2(x + 2)} \\ \equiv \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 2} \end{aligned}$$

and multiply through:

$$\begin{aligned} 4x^2 - 3x + 5 \\ \equiv A(x - 1)(x + 2) + B(x + 2) \\ + C(x - 1)^2 \end{aligned}$$

2. Substitute $x = 1$: $6 = 3B$, so $B = 2$. Substitute $x = -2$: $27 = 9C$, so $C = 3$.
 3. Compare the coefficients of x^2 : $4 = A + C$, so $A = 1$.
 4. So

$$\begin{aligned} \frac{4x^2 - 3x + 5}{(x - 1)^2(x + 2)} \\ \equiv \frac{1}{x - 1} + \frac{2}{(x - 1)^2} + \frac{3}{x + 2} \end{aligned}$$

COMMON MISTAKE

The x^2 coefficient on the right comes from **both** $A(x - 1)(x + 2)$ and $C(x - 1)^2$. Forgetting the C term gives $A = 4$, which is wrong. Check your answer by putting in a value such as $x = 0$: both sides here give $\frac{5}{2}$.

5. Improper fractions

A fraction is **improper** when the degree of the numerator is at least the degree of the denominator, like $\frac{x^2 + 3x + 5}{x + 1}$. Just as $\frac{17}{5} = 3 + \frac{2}{5}$, an improper algebraic fraction is a polynomial plus a proper fraction.

KEY RESULT

If the degree of $f(x)$ is at least the degree of $g(x)$, **divide first**:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

where $q(x)$ is the quotient and $r(x)$ is the remainder, whose degree is lower than that of $g(x)$. Then split the proper fraction that is left into partial fractions.

WORKED EXAMPLE 2.6 • DIVIDING FIRST

Express $\frac{x^2 + 3x + 5}{x + 1}$ in the form $Ax + B + \frac{C}{x + 1}$.

Divide $x^2 + 3x + 5$ by $(x + 1)$:

$$\begin{array}{r}
 \begin{array}{r} x \quad +2 \\ (x+1) \overline{) \begin{array}{r} x^2 \quad +3x \quad +5 \\ -(x^2 \quad +x) \\ \hline 2x \quad +5 \\ -(2x \quad +2) \\ \hline 3 \end{array}} \end{array}
 \end{array}
 \begin{array}{l}
 x(x+1) = x^2 + x \\
 2(x+1) = 2x + 2 \\
 \text{remainder}
 \end{array}$$

The quotient $x + 2$ builds along the top; the remainder is 3.

The quotient is $x + 2$ and the remainder is 3, so

$$\frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1}$$

WORKED EXAMPLE 2.7 • IMPROPER, THEN PARTIAL FRACTIONS

Express $\frac{3x^2 - 5x + 4}{(x - 1)(x - 2)}$ in partial fractions.

- Both top and bottom have degree 2, so divide first. The denominator is $x^2 - 3x + 2$, and taking

3 lots of it from the numerator leaves a linear remainder:

$$\begin{aligned} 3x^2 - 5x + 4 &= 3(x^2 - 3x + 2) \\ &\quad + (4x - 2) \end{aligned}$$

So the fraction is $3 + \frac{4x - 2}{(x - 1)(x - 2)}$.

2. Split the proper part: $4x - 2 \equiv A(x - 2) + B(x - 1)$. Put $x = 2$: $B = 6$. Put $x = 1$: $2 = -A$, so $A = -2$.

3. So

$$\begin{aligned} \frac{3x^2 - 5x + 4}{(x - 1)(x - 2)} \\ \equiv 3 - \frac{2}{x - 1} + \frac{6}{x - 2} \end{aligned}$$

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[3 marks]

Simplify fully:

(a) $\frac{x^2 - 16}{x^2 + 4x}$

(b) $\frac{2x^2 + 5x - 3}{4x^2 - 1}$

SOLUTION

(a)

$$\begin{aligned} \frac{x^2 - 16}{x^2 + 4x} &= \frac{(x - 4)(x + 4)}{x(x + 4)} \\ &= \frac{x - 4}{x} \end{aligned}$$

(b)

$$\begin{aligned}\frac{2x^2 + 5x - 3}{4x^2 - 1} &= \frac{(2x - 1)(x + 3)}{(2x - 1)(2x + 1)} \\ &= \frac{x + 3}{2x + 1}\end{aligned}$$

Question 2

[3 marks]

Write $\frac{2}{x+3} + \frac{5}{x-1}$ as a single fraction.

SOLUTION

Over the common denominator $(x+3)(x-1)$:

$$\begin{aligned}\frac{2(x-1) + 5(x+3)}{(x+3)(x-1)} \\ = \frac{7x+13}{(x+3)(x-1)}\end{aligned}$$

Question 3

[3 marks]

Simplify $\frac{x^2+2x}{x^2-1} \div \frac{x}{x+1}$.

SOLUTION

Multiply by the reciprocal and factorise:

$$\frac{x(x+2)}{(x-1)(x+1)} \times \frac{x+1}{x}$$

Cancelling x and $(x+1)$ leaves $\frac{x+2}{x-1}$.

Question 4

[3 marks]

Express $\frac{3x+7}{(x+1)(x+3)}$ in partial fractions.

SOLUTION

$$3x + 7 \equiv A(x+3) + B(x+1).$$

Put $x = -1$: $4 = 2A$, so $A = 2$. Put $x = -3$: $-2 = -2B$, so $B = 1$.

$$\frac{3x+7}{(x+1)(x+3)} \equiv \frac{2}{x+1} + \frac{1}{x+3}$$

Standard

Question 5

[4 marks]

Express $\frac{x-17}{(2x+1)(x-3)}$ in partial fractions.

SOLUTION

$$x-17 \equiv A(x-3) + B(2x+1).$$

Put $x = 3$: $-14 = 7B$, so $B = -2$.

Put $x = -\frac{1}{2}$: $-\frac{35}{2} = -\frac{7}{2}A$, so $A = 5$.

$$\frac{x-17}{(2x+1)(x-3)} \equiv \frac{5}{2x+1} - \frac{2}{x-3}$$

Question 6

[4 marks]

Express $\frac{2x^2+5x-4}{x(x-1)(x+2)}$ in partial fractions.

SOLUTION

$$\begin{aligned} 2x^2+5x-4 & \\ & \equiv A(x-1)(x+2) + Bx(x+2) \\ & \quad + Cx(x-1) \end{aligned}$$

Put $x = 0$: $-4 = -2A$, so $A = 2$.

Put $x = 1$: $3 = 3B$, so $B = 1$.

Put $x = -2$: $-6 = 6C$, so $C = -1$.

$$\begin{aligned} \frac{2x^2+5x-4}{x(x-1)(x+2)} & \\ & \equiv \frac{2}{x} + \frac{1}{x-1} - \frac{1}{x+2} \end{aligned}$$

Question 7

[4 marks]

Express $\frac{3x^2 + 2x + 2}{(x + 1)^2(x - 2)}$ in partial fractions.

SOLUTION

$$\begin{aligned} 3x^2 + 2x + 2 \\ \equiv A(x + 1)(x - 2) + B(x - 2) \\ + C(x + 1)^2 \end{aligned}$$

Put $x = -1$: $3 = -3B$, so $B = -1$.

Put $x = 2$: $18 = 9C$, so $C = 2$.

Compare the coefficients of x^2 : $3 = A + C$, so $A = 1$.

$$\begin{aligned} \frac{3x^2 + 2x + 2}{(x + 1)^2(x - 2)} \\ \equiv \frac{1}{x + 1} - \frac{1}{(x + 1)^2} + \frac{2}{x - 2} \end{aligned}$$

Question 8

[4 marks]

Express $\frac{2x^2 + 3x - 1}{x + 2}$ in the form $Ax + B + \frac{C}{x + 2}$.

SOLUTION

Instead of dividing, you can write the identity the answer must satisfy and compare coefficients:

$$\begin{aligned} 2x^2 + 3x - 1 &\equiv (Ax + B)(x + 2) \\ &+ C \end{aligned}$$

Put $x = -2$: $8 - 6 - 1 = C$, so $C = 1$.

Compare the coefficients of x^2 : $A = 2$.

Compare the constants: $-1 = 2B + C$, so $B = -1$.

$$\frac{2x^2 + 3x - 1}{x + 2} = 2x - 1 + \frac{1}{x + 2}$$

Question 9

[3 marks]

Show that $\frac{x}{x^2 - 4} - \frac{1}{x + 2}$ can be written as $\frac{k}{x^2 - 4}$, and find the value of k .

SOLUTION

$x^2 - 4 = (x - 2)(x + 2)$ already contains $(x + 2)$, so it is the common denominator:

$$\begin{aligned} \frac{x}{(x - 2)(x + 2)} - \frac{x - 2}{(x - 2)(x + 2)} \\ = \frac{x - (x - 2)}{x^2 - 4} = \frac{2}{x^2 - 4} \end{aligned}$$

So $k = 2$.

Exam style**Question 10**

[6 marks]

$$f(x) = \frac{2x^2 + 2x - 3}{(x + 1)(x - 2)}, \quad x > 2$$

- (a) Express $f(x)$ in the form $C + \frac{A}{x + 1} + \frac{B}{x - 2}$, where A , B and C are constants. [4]
 (b) Hence explain why $f(x) > 2$ for all $x > 2$. [2]

SOLUTION

(a) The fraction is improper, so divide first. The denominator is $x^2 - x - 2$, and

$$\begin{aligned} 2x^2 + 2x - 3 &= 2(x^2 - x - 2) \\ &\quad + (4x + 1) \end{aligned}$$

so $C = 2$. Then $4x + 1 \equiv A(x - 2) + B(x + 1)$.

Put $x = 2$: $9 = 3B$, so $B = 3$. Put $x = -1$: $-3 = -3A$, so $A = 1$.

$$f(x) = 2 + \frac{1}{x + 1} + \frac{3}{x - 2}$$

(b) For $x > 2$, both $x + 1$ and $x - 2$ are positive, so $\frac{1}{x + 1} > 0$ and $\frac{3}{x - 2} > 0$. Adding two positive terms to 2 gives more than 2.

Question 11

[5 marks]

- (a) Express $\frac{(x-1)^2}{x^2(2x+1)}$ in partial fractions. [4]
- (b) Check your answer by substituting $x = 1$ into both sides. [1]

SOLUTION

(a) The factor x^2 is repeated, so

$$\frac{(x-1)^2}{x^2(2x+1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+1}$$

$$(x-1)^2 \equiv Ax(2x+1) + B(2x+1) + Cx^2$$

Put $x = 0$: $1 = B$.

Put $x = -\frac{1}{2}$: $\frac{9}{4} = \frac{1}{4}C$, so $C = 9$.

Compare the coefficients of x^2 : $1 = 2A + C$, so $A = -4$.

$$\frac{(x-1)^2}{x^2(2x+1)} \equiv -\frac{4}{x} + \frac{1}{x^2} + \frac{9}{2x+1}$$

(b) At $x = 1$ the left side is 0, and the right side is $-4 + 1 + 3 = 0$. ✓

Question 12

[7 marks]

- (a) Use the factor theorem to show that $(x-1)$ is a factor of $x^3 - 7x + 6$, and hence factorise $x^3 - 7x + 6$ fully. [3]
- (b) Hence express $\frac{2x^2 + 11x - 25}{x^3 - 7x + 6}$ in partial fractions. [4]

SOLUTION

(a) $1^3 - 7(1) + 6 = 0$, so $(x-1)$ is a factor. Write $x^3 - 7x + 6 = (x-1)(x^2 + bx + c)$ and compare coefficients: the x^2 terms give $b - 1 = 0$, so $b = 1$, and the constants give $-c = 6$, so $c = -6$. Then

$$\begin{aligned} x^3 - 7x + 6 &= (x-1)(x^2 + x - 6) \\ &= (x-1)(x-2)(x+3) \end{aligned}$$

(b)

$$\begin{aligned}2x^2 + 11x - 25 \\ \equiv A(x - 2)(x + 3) \\ + B(x - 1)(x + 3) \\ + C(x - 1)(x - 2)\end{aligned}$$

Put $x = 1$: $-12 = -4A$, so $A = 3$.

Put $x = 2$: $5 = 5B$, so $B = 1$.

Put $x = -3$: $-40 = 20C$, so $C = -2$.

$$\begin{aligned}\frac{2x^2 + 11x - 25}{x^3 - 7x + 6} \\ \equiv \frac{3}{x - 1} + \frac{1}{x - 2} - \frac{2}{x + 3}\end{aligned}$$