

YEAR 2 PURE • CHAPTER 11

Parametric Equations

Specification: Edexcel Pure 3.3, 3.4 — Coordinate geometry • AQA section C3, C4

KEY FACTS — THIS CHAPTER

- A **parametric** curve gives x and y separately in terms of a third variable, the **parameter**, often t or θ .
- Each value of the parameter gives one point (x, y) . As the parameter changes, the point traces the curve.
- To find the **Cartesian** equation, eliminate the parameter: make it the subject of one equation and substitute, or use an identity such as $\sin^2 t + \cos^2 t \equiv 1$.
- The values x can take are the range of $x(t)$, which may **restrict** the Cartesian curve to only part of its graph.
- To find where a curve meets a line, substitute $x(t)$ and $y(t)$ into the line's equation and solve for t .

A ball in flight has a horizontal position and a height, and both change with time. Writing each in terms of time t describes its path, and says where the ball is at every moment, which an equation linking x and y alone can't do.

1. Curves traced by a parameter

KEY RESULT

A **parametric curve** is given by two equations

$$x = f(t), \quad y = g(t)$$

with the **parameter** t taking values in some interval. Each value of t gives one point on the curve.

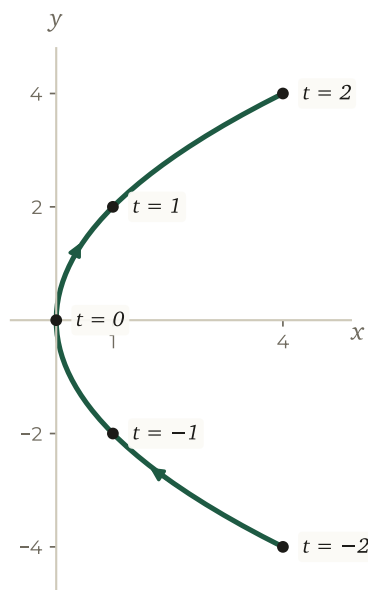
WORKED EXAMPLE 11.1 • PLOTTING FROM A TABLE

Sketch the curve $x = t^2$, $y = 2t$ for $-2 \leq t \leq 2$.

Work out a point for each of a few values of t :

t	-2	-1	0	1	2
$x = t^2$	4	1	0	1	4
$y = 2t$	-4	-2	0	2	4

Plot the points in order of t and join them with a smooth curve. It is a parabola on its side, traced from $(4, -4)$ up to $(4, 4)$.



$x = t^2$, $y = 2t$, traced as t increases from -2 to 2.

2. Finding the Cartesian equation

To find an equation linking x and y alone, eliminate the parameter.

WORKED EXAMPLE 11.2 • ELIMINATING THE PARAMETER

Find the Cartesian equation of the curve $x = t + 1$, $y = t^2 - 3$, for $-2 \leq t \leq 2$, and state its domain.

1. Make t the subject of the simpler equation: $t = x - 1$.
2. Substitute into the other:

$$y = (x - 1)^2 - 3$$

3. As t runs from -2 to 2 , $x = t + 1$ runs from -1 to 3 . So the curve is only the part of this parabola with $-1 \leq x \leq 3$.

WORKED EXAMPLE 11.3 • A CIRCLE

Show that the curve $x = 2 + 3 \cos t$, $y = -1 + 3 \sin t$, $0 \leq t < 2\pi$, is a circle, and give its centre and radius.

1. Isolate the trig functions: $\cos t = \frac{x-2}{3}$ and $\sin t = \frac{y+1}{3}$.
2. Use $\cos^2 t + \sin^2 t \equiv 1$:

$$\begin{aligned} \frac{(x-2)^2}{9} + \frac{(y+1)^2}{9} &= 1 \\ (x-2)^2 + (y+1)^2 &= 9 \end{aligned}$$

So it is the circle with centre $(2, -1)$ and radius 3.

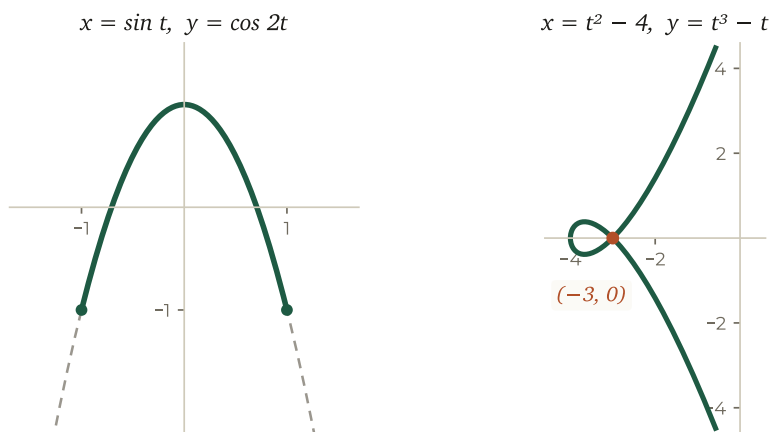
WORKED EXAMPLE 11.4 • A RESTRICTED CURVE

Find the Cartesian equation of the curve $x = \sin t$, $y = \cos 2t$, and state the values of x for which it is defined.

1. Use the double-angle formula that contains only $\sin t$:

$$\begin{aligned} y = \cos 2t &= 1 - 2 \sin^2 t \\ &= 1 - 2x^2 \end{aligned}$$

2. $x = \sin t$ can only take values from -1 to 1 . So the curve is the part of the parabola $y = 1 - 2x^2$ with $-1 \leq x \leq 1$. It is traced backwards and forwards along this arc as t increases.



Left: $x = \sin t$, $y = \cos 2t$ is only the arc of $y = 1 - 2x^2$ with $-1 \leq x \leq 1$. Right: $x = t^2 - 4$, $y = t^3 - t$ crosses itself at $(-3, 0)$.

COMMON MISTAKE

The Cartesian equation can describe more than the curve does. Always check which values x and y can take: in Example 11.4, $y = 1 - 2x^2$ goes on for ever, but the parametric curve stops at $x = \pm 1$.

3. Intersections

To find where a parametric curve meets a line or another curve, substitute $x(t)$ and $y(t)$ into the other equation. That gives an equation in t alone. Solve it, then turn each value of t back into a point.

WORKED EXAMPLE 11.5 • MEETING A LINE

Find the points where the curve $x = t^2$, $y = 2t$ meets the line $y = x - 3$.

1. Substitute into the line:

$$\begin{aligned} 2t &= t^2 - 3 \\ t^2 - 2t - 3 &= 0 \\ (t - 3)(t + 1) &= 0 \end{aligned}$$

2. $t = 3$ gives the point $(9, 6)$, and $t = -1$ gives $(1, -2)$.

WORKED EXAMPLE 11.6 • MEETING THE AXES

The curve C is $x = t^2 - 4$, $y = t^3 - t$. Find where C meets the coordinate axes.

1. On the y -axis, $x = 0$, so $t^2 = 4$ and $t = \pm 2$. Then $y = 8 - 2 = 6$ or $y = -8 + 2 = -6$: the points $(0, 6)$ and $(0, -6)$.
2. On the x -axis, $y = 0$, so $t(t^2 - 1) = 0$ and $t = 0$ or ± 1 . Then $t = 0$ gives $(-4, 0)$, and both $t = 1$ and $t = -1$ give $(-3, 0)$.

Two values of t give the same point, so the curve passes through $(-3, 0)$ twice: it crosses itself there, as the figure above shows.

4. Modelling with parametric equations

WORKED EXAMPLE 11.7 • A PROJECTILE

A ball is kicked from the ground. After t seconds it is $x = 20t$ metres along the ground and $y = 15t - 4.9t^2$ metres high. Find how far it travels before it lands, its greatest height, and the Cartesian equation of its path.

1. It lands when $y = 0$: $t(15 - 4.9t) = 0$, so $t = \frac{15}{4.9} = 3.061 \dots$. It is then $x = 20 \times 3.061 \dots = 61.2$ m away (3 s.f.).
2. y is a quadratic in t , greatest halfway between its roots, at $t = \frac{15}{9.8} = 1.530 \dots$:

$$\begin{aligned} y &= 15 \times 1.530 \dots \\ &\quad - 4.9 \times 1.530 \dots^2 \\ &= 11.5 \text{ m (3 s.f.)} \end{aligned}$$

3. $t = \frac{x}{20}$, so

$$\begin{aligned} y &= 15 \times \frac{x}{20} - 4.9 \times \frac{x^2}{400} \\ &= 0.75x - 0.01225x^2 \end{aligned}$$

for $0 \leq x \leq 61.2$.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[2 marks]

Find the Cartesian equation of the curve $x = 2t - 1$, $y = t^2 + 1$.

SOLUTION

$t = \frac{x+1}{2}$, so

$$y = \left(\frac{x+1}{2}\right)^2 + 1 = \frac{(x+1)^2}{4} + 1$$

Question 2

[2 marks]

Describe the curve $x = 4 \cos \theta$, $y = 4 \sin \theta$, $0 \leq \theta < 2\pi$.

SOLUTION

$x^2 + y^2 = 16 \cos^2 \theta + 16 \sin^2 \theta = 16$, so the curve is the circle with centre $(0, 0)$ and radius 4.

Question 3

[2 marks]

A curve is given by $x = t^3$, $y = t^2 - 1$.

- (a) Find the point where $t = 2$. [1]
- (b) Find the value of t at the point $(-1, 0)$. [1]

SOLUTION

(a) $x = 8$ and $y = 3$, so the point is $(8, 3)$.

(b) $t^3 = -1$ gives $t = -1$, and then $y = 1 - 1 = 0$ as required.

Question 4

[2 marks]

Find the Cartesian equation of the curve $x = \frac{1}{t}$, $y = t + 1$, where $t \neq 0$.

SOLUTION

$$t = \frac{1}{x}, \text{ so } y = \frac{1}{x} + 1.$$

Standard**Question 5**

[3 marks]

A curve is given by $x = 3 + 2 \cos t$, $y = -1 + 2 \sin t$, $0 \leq t < 2\pi$. Find its Cartesian equation, and describe the curve.

SOLUTION

$$\cos t = \frac{x-3}{2} \text{ and } \sin t = \frac{y+1}{2}. \text{ Using } \cos^2 t + \sin^2 t \equiv 1:$$

$$(x - 3)^2 + (y + 1)^2 = 4$$

It is the circle with centre $(3, -1)$ and radius 2.

Question 6

[4 marks]

A curve is given by $x = \cos t$, $y = \sin 2t$. Show that its Cartesian equation is $y^2 = 4x^2(1 - x^2)$.

SOLUTION

$$y = \sin 2t = 2 \sin t \cos t. \text{ Squaring removes the sign of } \sin t:$$

$$\begin{aligned} y^2 &= 4 \sin^2 t \cos^2 t \\ &= 4(1 - \cos^2 t) \cos^2 t \\ &= 4(1 - x^2)x^2 \end{aligned}$$

Question 7

[4 marks]

A curve is given by $x = 2 \sin t$, $y = \cos 2t$.

- (a) Find its Cartesian equation. [2]
- (b) State the values that x and y can take. [2]

SOLUTION

(a) $\cos 2t = 1 - 2 \sin^2 t$ and $\sin t = \frac{x}{2}$, so

$$y = 1 - 2 \times \frac{x^2}{4} = 1 - \frac{x^2}{2}$$

(b) $x = 2 \sin t$ takes values $-2 \leq x \leq 2$, and $y = \cos 2t$ takes values $-1 \leq y \leq 1$.

Question 8

[4 marks]

Find the points where the curve $x = t^2$, $y = t^3$ meets the line $y = 2x$.

SOLUTION

Substitute into the line:

$$\begin{aligned} t^3 &= 2t^2 \\ t^2(t - 2) &= 0 \end{aligned}$$

$t = 0$ gives $(0, 0)$, and $t = 2$ gives $(4, 8)$.

Question 9

[4 marks]

Find the coordinates of the points where the curve $x = t^2 - 1$, $y = t(t - 2)$ meets the coordinate axes.

SOLUTION

On the y -axis, $x = 0$, so $t = \pm 1$. Then $t = 1$ gives $y = -1$ and $t = -1$ gives $y = 3$: the points $(0, -1)$ and $(0, 3)$.

On the x -axis, $y = 0$, so $t = 0$ or $t = 2$. These give $(-1, 0)$ and $(3, 0)$.

Exam style**Question 10**

[8 marks]

A ball is thrown from a window. Its position t seconds later is modelled by $x = 12t$ and $y = 1.5 + 9t - 4.9t^2$, where x and y are in metres, measured along the ground and up from the ground.

- (a) State the height from which the ball is thrown. [1]
- (b) Find the time at which the ball hits the ground, and how far it has then travelled horizontally, to 3 significant figures. [4]
- (c) Find the Cartesian equation of its path. [2]

(d) Give one way in which the model could be improved. [1]

SOLUTION

(a) When $t = 0$, $y = 1.5$ m.

(b) It hits the ground when $4.9t^2 - 9t - 1.5 = 0$. By the quadratic formula, taking the positive root:

$$\begin{aligned} t &= \frac{9 + \sqrt{81 + 29.4}}{9.8} \\ &= 1.99 \text{ s (3 s.f.)} \end{aligned}$$

It has then travelled $x = 12 \times 1.990 \dots = 23.9$ m (3 s.f.).

(c) $t = \frac{x}{12}$, so

$$\begin{aligned} y &= 1.5 + 9 \times \frac{x}{12} - 4.9 \times \frac{x^2}{144} \\ &= 1.5 + 0.75x - \frac{49x^2}{1440} \end{aligned}$$

(d) For example, include air resistance, which the model ignores.

Question 11

[6 marks]

A curve is given by $x = \tan t$, $y = \sec t$, for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

(a) Show that the Cartesian equation is $y = \sqrt{1 + x^2}$. [3]

(b) Find the points where the curve meets the line $y = 2$. [3]

SOLUTION

(a) $\sec^2 t = 1 + \tan^2 t$, so $y^2 = 1 + x^2$. For $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $\cos t > 0$, so $y = \sec t > 0$ and $y = \sqrt{1 + x^2}$.

(b) $\sec t = 2$ means $\cos t = \frac{1}{2}$, so $t = \pm\frac{\pi}{3}$. Then $x = \tan(\pm\frac{\pi}{3}) = \pm\sqrt{3}$: the points are $(\sqrt{3}, 2)$ and $(-\sqrt{3}, 2)$.

Question 12

[7 marks]

The curve C is given by $x = 1 + 2 \cos t$, $y = 2 + 2 \sin t$, $0 \leq t < 2\pi$. The line $y = x + 1$ meets C at A and B .

(a) Show that at A and B , $\tan t = 1$. [2]

(b) Find the exact coordinates of A and B . [3]

(c) Show that AB is a diameter of C , and state its length. [2]

SOLUTION

(a) Substitute into the line: $2 + 2 \sin t = 1 + 2 \cos t + 1$, so $\sin t = \cos t$, and dividing by $\cos t$ gives $\tan t = 1$.

(b) $t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. With $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, the points are $(1 + \sqrt{2}, 2 + \sqrt{2})$ and $(1 - \sqrt{2}, 2 - \sqrt{2})$.

(c) C is the circle with centre $(1, 2)$ and radius 2. The midpoint of AB is $(1, 2)$, the centre, so AB is a diameter, of length 4.