

YEAR 2 PURE • CHAPTER 14

Numerical Methods

Specification: Edexcel Pure 9.1–9.5 — Numerical methods • AQA section 11–14

KEY FACTS — THIS CHAPTER

- If f is **continuous** on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has a root between a and b .
- A change of sign can mislead: an asymptote gives a sign change with no root, and a repeated root or two close roots can give none.
- **Iteration:** rearrange $f(x) = 0$ as $x = g(x)$ and use $x_{n+1} = g(x_n)$. The steps make a **staircase** or a **cobweb**, and some rearrangements diverge.
- **Newton–Raphson:** $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. It is fast, but fails if $f'(x_n) = 0$ and can jump to another root if $f'(x_n)$ is small.
- **Trapezium rule:** $\int_a^b y \, dx \approx \frac{h}{2} [y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n]$, with $h = \frac{b-a}{n}$.
- It **overestimates** where the curve is convex and **underestimates** where it is concave.

Most equations can't be solved exactly. $x^3 - 2x - 5 = 0$ has one real root, and no rearranging will produce a neat formula for it. Numerical methods find such roots, and the areas under curves you can't integrate, to any accuracy you need: each step gets closer, and you stop when the answer is good enough.

1. Locating roots by a change of sign

KEY RESULT

If $f(x)$ is continuous on the interval $[a, b]$, and $f(a)$ and $f(b)$ have opposite signs, then there is at least one root of $f(x) = 0$ between a and b .

The graph of a continuous function has no gaps, so to get from below the axis to above it, it has to cross the axis somewhere in between.

WORKED EXAMPLE 14.1 · LOCATING AND REFINING A ROOT

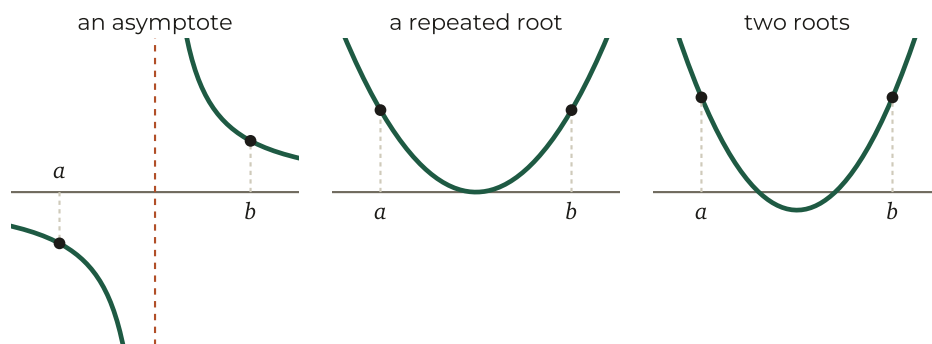
Show that $x^3 - 2x - 5 = 0$ has a root between 2 and 3, and show that it is 2.09 to 2 decimal places.

1. Let $f(x) = x^3 - 2x - 5$. Then $f(2) = -1 < 0$ and $f(3) = 16 > 0$. f is a polynomial, so it is continuous, and the change of sign shows there is a root between 2 and 3.
2. To show the root is 2.09 to 2 d.p., test either side of the interval that rounds to 2.09:

$$f(2.085) = -0.106 \text{ (3 s.f.)} < 0$$

$$f(2.095) = 0.00501 \text{ (3 s.f.)} > 0$$

3. There is a change of sign, so the root lies between 2.085 and 2.095, and is 2.09 to 2 decimal places.



Three ways a change of sign misleads: an asymptote (left), a repeated root (middle) and two roots in one interval (right).

COMMON MISTAKE

Check that the function is continuous on the interval. $f(x) = \frac{1}{x-2}$ changes sign between 1 and 3, but has no root there: the sign change comes from the asymptote at $x = 2$.

2. Iteration

To solve $f(x) = 0$ by iteration, rearrange it into the form $x = g(x)$, pick a starting value x_0 , and use the rule over and over:

KEY RESULT

$$x_{n+1} = g(x_n)$$

If the values settle down, they settle on a root, where the line $y = x$ meets the curve $y = g(x)$. On a diagram the steps form a **staircase** when they approach the root from one side, and a **cobweb** when they spiral round it.

WORKED EXAMPLE 14.2 • A REARRANGEMENT THAT WORKS

Use the iteration $x_{n+1} = \sqrt[3]{2x_n + 5}$ with $x_0 = 2$ to find the root of $x^3 - 2x - 5 = 0$.

1. $x^3 = 2x + 5$, so the root satisfies $x = \sqrt[3]{2x + 5}$.
2. Iterate, keeping the calculator's full value each time:

$$x_1 = \sqrt[3]{9} = 2.0801$$

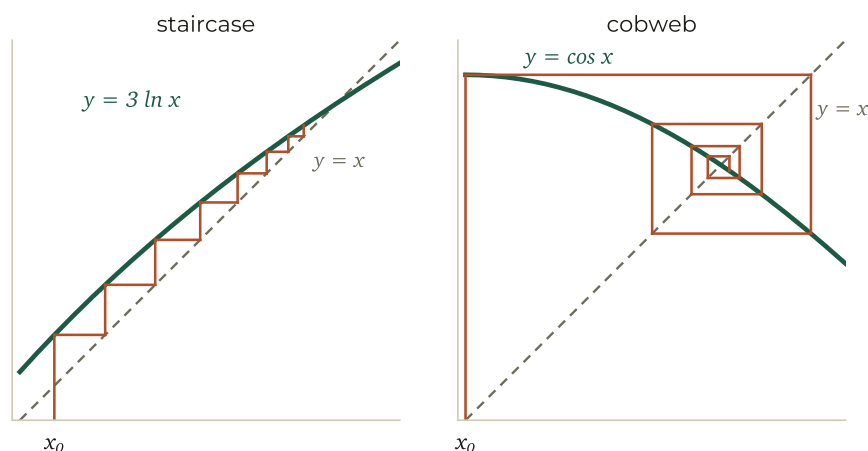
$$x_2 = 2.0924$$

$$x_3 = 2.0942$$

$$x_4 = 2.0945$$

(all to 4 d.p.).

3. The values climb steadily and settle, so the root is 2.09 to 2 decimal places, agreeing with Example 14.1.



A staircase for $x = 3 \ln x$ (left) and a cobweb for $x = \cos x$ (right). Both close in on the root where the curve meets $y = x$.

WORKED EXAMPLE 14.3 · A REARRANGEMENT THAT FAILS

Show that the rearrangement $x_{n+1} = \frac{x_n^3 - 5}{2}$ of the same equation does not find the root, starting from $x_0 = 2$.

1. Iterate: $x_1 = 1.5$, $x_2 = -0.8125$, $x_3 = -2.768$, $x_4 = -13.11$ (4 s.f.).
2. The values run further and further from the root at about 2.09: this iteration **diverges**.

WHICH REARRANGEMENTS WORK

Near the root, each step multiplies the error by about $g'(x)$. So the iteration closes in when $|g'(x)| < 1$ near the root, and moves away when $|g'(x)| > 1$. At the root, $g(x) = \sqrt[3]{2x+5}$ has $g'(x) \approx 0.15$, but $g(x) = \frac{x^3-5}{2}$ has $g'(x) = \frac{3x^2}{2} \approx 6.6$. A positive g' gives a staircase, and a negative one gives a cobweb.

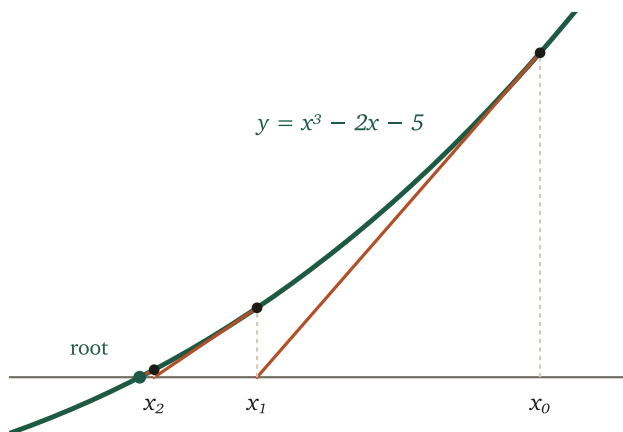
3. The Newton–Raphson method

Newton–Raphson follows the tangent. From a point on the curve, slide down the tangent to where it meets the x -axis. That is usually much closer to the root, so repeat from there.

KEY RESULT

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The tangent at $(x_n, f(x_n))$ has gradient $f'(x_n)$, so it meets the axis after a horizontal distance of $\frac{f(x_n)}{f'(x_n)}$.



Newton-Raphson for $y = x^3 - 2x - 5$, starting from $x_0 = 3$: each tangent lands closer to the root.

WORKED EXAMPLE 14.4 • NEWTON-RAPHSON

Use the Newton-Raphson method with $x_0 = 2$ to find the root of $x^3 - 2x - 5 = 0$ to 6 decimal places.

1. $f(x) = x^3 - 2x - 5$ and $f'(x) = 3x^2 - 2$, so

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

2. $x_1 = 2 - \frac{-1}{10} = 2.1$, then $x_2 = 2.094568$ and $x_3 = 2.094551$ (6 d.p.).
3. $x_4 = 2.094551$ too, so the root is 2.094551 to 6 decimal places.

The number of correct digits roughly doubles at each step, far faster than the iteration in Example 14.2.

COMMON MISTAKE

Newton–Raphson fails if $f'(x_n) = 0$: the tangent is horizontal and never meets the axis. For $f(x) = x^3 - 3x + 1$, starting at $x_0 = 1$ gives $f'(1) = 0$. A start **near** a stationary point is almost as bad: the tangent is nearly flat, so the next value lands far away, perhaps near a different root.

4. The trapezium rule

To estimate $\int_a^b y \, dx$, split the area into n strips of width h and treat each strip as a trapezium.

KEY RESULT

With $h = \frac{b-a}{n}$ and $y_0, y_1, \dots, y_n$ the heights at the $n+1$ equally spaced x -values,

$$\int_a^b y \, dx \approx \frac{h}{2} [y_0 + 2(y_1 + \dots + y_{n-1}) + y_n]$$

The first and last heights are counted once and the rest twice, because each inner height is shared by two trapezia. More strips give a better estimate.

WORKED EXAMPLE 14.5 • ESTIMATING AN AREA

Use the trapezium rule with 4 strips to estimate $\int_1^3 \ln x \, dx$. State whether it is an overestimate or an underestimate.

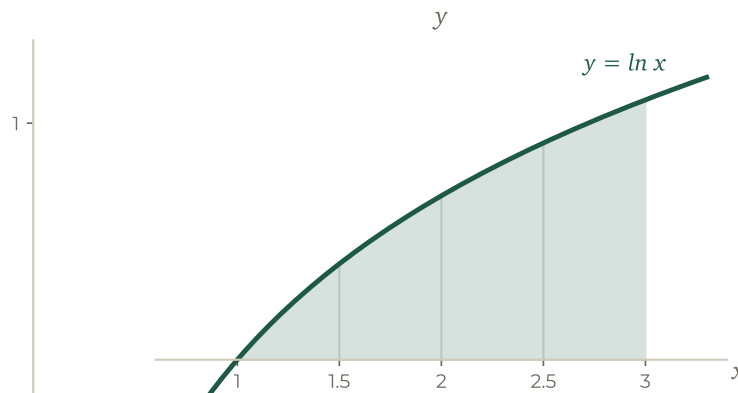
$h = \frac{3-1}{4} = 0.5$, so the heights are at $x = 1, 1.5, 2, 2.5, 3$:

x	$\ln x$
1	0
1.5	0.405465
2	0.693147
2.5	0.916291
3	1.098612

Apply the rule:

$$\begin{aligned} & \frac{0.5}{2} [0 + 2(0.405465 + 0.693147 \\ & \quad + 0.916291) + 1.098612] \\ & = 1.2821 \text{ (4 d.p.)} \end{aligned}$$

$y = \ln x$ is concave, so each trapezium's slanted top lies below the curve, and 1.2821 is an **underestimate**. The exact value, found by a method in Chapter 15, is $3 \ln 3 - 2 = 1.2958$ (4 d.p.).



Four trapezia under $y = \ln x$. The curve is concave, so the trapezia miss a sliver of area and underestimate the integral.

OVER OR UNDER?

- Where the curve is **convex**, the tops of the trapezia lie above it, so the rule **overestimates**.
- Where the curve is **concave**, they lie below it, so the rule **underestimates**.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[2 marks]

Show that $e^x - 3x = 0$ has a root between $x = 1$ and $x = 2$.

SOLUTION

Let $f(x) = e^x - 3x$. Then $f(1) = e - 3 = -0.282 < 0$ and $f(2) = e^2 - 6 = 1.39 > 0$ (3 s.f.). f is continuous, so the change of sign shows there is a root between 1 and 2.

Question 2

[2 marks]

$\tan 1 > 0$ and $\tan 2 < 0$. Explain why this does not show that $\tan x = 0$ has a root between 1 and 2.

SOLUTION

$\tan x$ is not continuous on $[1, 2]$: it has an asymptote at $x = \frac{\pi}{2} \approx 1.57$. The change of sign comes from the asymptote, not a root.

Question 3

[2 marks]

Use the iteration $x_{n+1} = \sqrt{x_n + 3}$ with $x_0 = 2$ to find x_1 , x_2 and x_3 to 4 decimal places.

SOLUTION

$x_1 = \sqrt{5} = 2.2361$, $x_2 = 2.2882$ and $x_3 = 2.2996$ (4 d.p.), keeping full calculator values between steps.

Question 4

[3 marks]

Use the trapezium rule with 4 strips to estimate $\int_0^2 x^2 dx$. State whether your answer is an overestimate or an underestimate, and explain why.

SOLUTION

$h = 0.5$, and the heights at $x = 0, 0.5, 1, 1.5, 2$ are 0, 0.25, 1, 2.25, 4:

$$\begin{aligned} & \frac{0.5}{2} [0 + 2(0.25 + 1 + 2.25) + 4] \\ &= 0.25 \times 11 = 2.75 \end{aligned}$$

$y = x^2$ is convex, so the trapezia lie above the curve and 2.75 is an overestimate. (The exact value is $\frac{8}{3}$.)

Standard**Question 5**

[4 marks]

(a) Show that $x^3 + x - 3 = 0$ has a root α between 1 and 2. [2]

(b) Show that $\alpha = 1.21$ to 2 decimal places. [2]

SOLUTION

(a) Let $f(x) = x^3 + x - 3$. $f(1) = -1 < 0$ and $f(2) = 7 > 0$. f is continuous, so there is a root between 1 and 2.

(b) $f(1.205) = -0.0453 < 0$ and $f(1.215) = 0.00861 > 0$ (3 s.f.). The change of sign shows $1.205 < \alpha < 1.215$, so $\alpha = 1.21$ to 2 decimal places.

Question 6

[4 marks]

Use the Newton–Raphson method with $x_0 = 1$ to find the root of $x^3 + x - 3 = 0$ to 4 decimal places.

SOLUTION

$f(x) = x^3 + x - 3$ and $f'(x) = 3x^2 + 1$, so

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 3}{3x_n^2 + 1}$$

$x_1 = 1 - \frac{-1}{4} = 1.25$, $x_2 = 1.2143$, $x_3 = 1.2134$ and $x_4 = 1.2134$ (4 d.p.). So the root is 1.2134 to 4 decimal places.

Question 7

[5 marks]

(a) Show that $x^3 + x - 3 = 0$ can be written as $x = \sqrt[3]{3 - x}$. [1]

(b) Use $x_{n+1} = \sqrt[3]{3 - x_n}$ with $x_0 = 1$ to find x_1 to x_4 , to 4 decimal places. [2]

(c) Say whether the iteration makes a staircase or a cobweb diagram, and why. [2]

SOLUTION

(a) $x^3 = 3 - x$, so $x = \sqrt[3]{3 - x}$.

(b) $x_1 = 1.2599$, $x_2 = 1.2028$, $x_3 = 1.2158$, $x_4 = 1.2129$ (4 d.p.).

(c) A cobweb: the values go above and below the root in turn, closing in from alternate sides.

Question 8

[5 marks]

(a) Use the trapezium rule with 4 strips to estimate $\int_0^1 \sqrt{1+x^2} dx$, to 4 decimal places. [3]

(b) Given that $\frac{d^2}{dx^2} \sqrt{1+x^2} = (1+x^2)^{-\frac{3}{2}}$, state whether your estimate is too big or too small. [2]

SOLUTION

(a) $h = 0.25$. The heights at $x = 0, 0.25, 0.5, 0.75, 1$ are 1, 1.030776, 1.118034, 1.25 and 1.414214, so the estimate is

$$\begin{aligned} \frac{0.25}{2} [1 + 2(1.030776 + 1.118034 \\ + 1.25) + 1.414214] \\ = 1.1515 \text{ (4 d.p.)} \end{aligned}$$

(b) The second derivative is positive, so the curve is convex and the estimate is too big.

Question 9

[4 marks]

$$f(x) = x^3 - 3x + 1.$$

- (a) Explain why the Newton–Raphson method fails with $x_0 = 1$. [2]
 (b) Using $x_0 = 0.9$, find x_1 , and comment on your answer. [2]

SOLUTION

(a) $f'(x) = 3x^2 - 3$, so $f'(1) = 0$. The formula would divide by zero: the tangent at $x = 1$ is horizontal and never meets the x -axis.

(b) $f(0.9) = -0.971$ and $f'(0.9) = -0.57$, so

$$x_1 = 0.9 - \frac{-0.971}{-0.57} = -0.804 \text{ (3 s.f.)}$$

The tangent is nearly flat, so x_1 lands far away, past the root near 0.35 and nowhere near the one near 1.53.

Exam style**Question 10**

[7 marks]

A town's population t years from now is modelled by $200e^{0.1t}$ thousand, and the number of homes it can support by $800 + 50t$ thousand. Let $f(t) = 200e^{0.1t} - 50t - 800$.

- (a) Show that $f(t) = 0$ has a root between $t = 20$ and $t = 25$. [2]
 (b) Use the Newton–Raphson method with $t_0 = 22$ to find this root to 2 decimal places. [4]
 (c) Interpret your answer. [1]

SOLUTION

(a) $f(20) = -322$ and $f(25) = 386$ (3 s.f.). f is continuous, so there is a root between 20 and 25.

(b) $f'(t) = 20e^{0.1t} - 50$, so

$$t_{n+1} = t_n - \frac{200e^{0.1t_n} - 50t_n - 800}{20e^{0.1t_n} - 50}$$

$t_1 = 22.7279$, $t_2 = 22.6939$ and $t_3 = 22.6939$ (4 d.p.), so the root is 22.69 to 2 decimal places.

(c) After about 22.7 years the population outgrows the homes available.

Question 11

[7 marks]

$x^2 - 5x + 3 = 0$ has two roots, near 0.70 and 4.30.

- (a) Show that the equation can be written as $x = \frac{x^2 + 3}{5}$ and as $x = 5 - \frac{3}{x}$. [2]
- (b) Starting from $x_0 = 1$, find x_1 , x_2 and x_3 for each rearrangement, to 4 decimal places. [3]
- (c) Explain why the two iterations lead to different roots. [2]

SOLUTION

(a) $5x = x^2 + 3$, so $x = \frac{x^2 + 3}{5}$. Dividing $x^2 = 5x - 3$ by x gives $x = 5 - \frac{3}{x}$.

(b) The first gives 0.8, 0.728, 0.7060 (4 d.p.), heading for the root near 0.70. The second gives 2, 3.5, 4.1429 (4 d.p.), heading for the root near 4.30.

(c) An iteration closes in on a root where $|g'(x)| < 1$. For $g(x) = \frac{x^2 + 3}{5}$, $g'(x) = \frac{2x}{5}$, which is about 0.28 at the small root but 1.7 at the large one. For $g(x) = 5 - \frac{3}{x}$, $g'(x) = \frac{3}{x^2}$, about 6.2 at the small root but 0.16 at the large one. So each iteration can only find one of the roots.

Question 12

[7 marks]

- (a) Use the trapezium rule with 4 strips to estimate $\int_1^2 \frac{1}{x} dx$, to 4 decimal places. [3]
- (b) Given that the exact value is $\ln 2$, find the percentage error in your estimate. [2]
- (c) Explain why the estimate is too big, and say how it could be improved. [2]

SOLUTION

(a) $h = 0.25$. The heights at $x = 1, 1.25, 1.5, 1.75, 2$ are 1, 0.8, 0.666667, 0.571429 and 0.5:

$$\begin{aligned} & \frac{0.25}{2} [1 + 2(0.8 + 0.666667 \\ & \quad + 0.571429) + 0.5] \\ & = 0.6970 \text{ (4 d.p.)} \end{aligned}$$

(b) $\ln 2 = 0.693147\dots$, so the percentage error is

$$\frac{0.697024 - 0.693147}{0.693147} \times 100 = 0.56\%$$

to 2 significant figures.

(c) $y = \frac{1}{x}$ is convex for $x > 0$, so the trapezia lie above the curve. More, narrower strips would give a better estimate.