

YEAR 2 PURE · CHAPTER 4

Modulus and Graph Transformations

Specification: Edexcel Pure 2.7, 2.9 — Algebra and functions · AQA section B7, B9

KEY FACTS — THIS CHAPTER

- The **modulus** $|x|$ is the size of x , ignoring its sign: $|x| = x$ when $x \geq 0$, and $|x| = -x$ when $x < 0$.
- $y = |ax + b|$ is a **V shape** with its vertex on the x -axis at $x = -\frac{b}{a}$.
- $|x - a| < b$ means x is within b of a : $a - b < x < a + b$.
- To solve $|f(x)| = g(x)$, solve $f(x) = g(x)$ **and** $-f(x) = g(x)$, then **check** each answer, since $g(x)$ can't be negative.
- $y = |f(x)|$: reflect the parts of $y = f(x)$ below the x -axis **in the x -axis**.
- $y = f(|x|)$: keep the part for $x \geq 0$ and reflect it **in the y -axis**.
- Combining transformations: apply them one at a time. Changes inside the bracket act on x , in reverse.

The modulus takes the sign off a number, so $|-5| = |5| = 5$. On a graph that is a fold: whatever dips below the axis is flipped back up. This chapter draws those folds, solves equations with them in, and finishes the Year 1 work on transformations by combining several at once.

1. The modulus function

KEY RESULT

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|x|$ is the **distance** of x from 0, and $|x - a|$ is the distance between x and a . The graph of $y = |ax + b|$ is the line $y = ax + b$ with its negative part reflected in the x -axis: a V with its vertex at $(-\frac{b}{a}, 0)$.

WORKED EXAMPLE 4.1 · A V-SHAPED GRAPH

Sketch $y = |2x - 3|$, giving the coordinates of the points where it meets the axes.

1. The vertex is where $2x - 3 = 0$, at $(\frac{3}{2}, 0)$.
2. When $x = 0$, $y = |-3| = 3$, so it meets the y -axis at $(0, 3)$.
3. To the right of the vertex the graph is $y = 2x - 3$. To the left it is $y = -(2x - 3) = 3 - 2x$. Both arms have gradient of size 2.

2. Modulus equations and inequalities

The equation $|f(x)| = g(x)$ says that $f(x)$ is either $g(x)$ or $-g(x)$. So there are two equations to solve, but only answers that make $g(x) \geq 0$ can work, because a modulus is never negative.

SOLVING MODULUS EQUATIONS

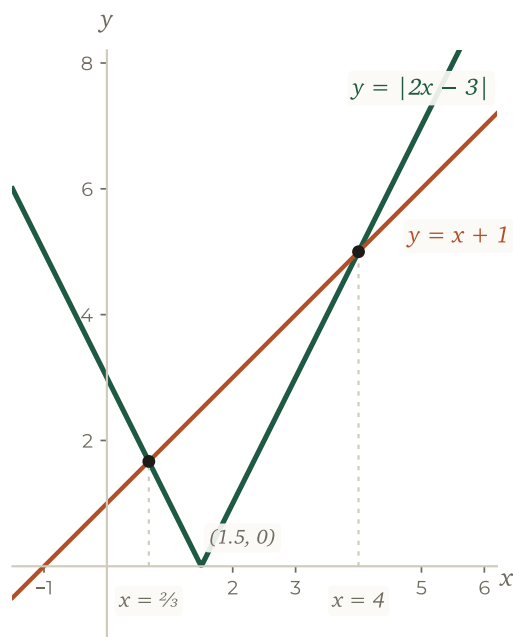
1. Solve $f(x) = g(x)$ and $-f(x) = g(x)$.
2. **Check** each solution in the original equation, rejecting any that make $g(x)$ negative.

A sketch of both graphs shows how many solutions to expect.

WORKED EXAMPLE 4.2 · A V MEETS A LINE TWICE

Solve $|2x - 3| = x + 1$.

1. First case, $2x - 3 = x + 1$: $x = 4$.
2. Second case, $-(2x - 3) = x + 1$, so $3 - 2x = x + 1$ and $x = \frac{2}{3}$.
3. Check: at $x = 4$ both sides are 5, and at $x = \frac{2}{3}$ both sides are $\frac{5}{3}$. Both solutions stand.



$y = |2x - 3|$ meets $y = x + 1$ at $x = \frac{2}{3}$ and $x = 4$. Between them the V is below the line.

WORKED EXAMPLE 4.3 · ONE ANSWER REJECTED

Solve $|x - 2| = 2x + 1$.

1. First case, $x - 2 = 2x + 1$: $x = -3$. But then $2x + 1 = -5$, and a modulus can't equal a negative number. Reject it.
2. Second case, $-(x - 2) = 2x + 1$, so $2 - x = 2x + 1$ and $x = \frac{1}{3}$. Check: $|\frac{1}{3} - 2| = \frac{5}{3}$ and $2(\frac{1}{3}) + 1 = \frac{5}{3}$.
3. The only solution is $x = \frac{1}{3}$.

COMMON MISTAKE

Always check your answers. The two cases produce candidates, not solutions: here $x = -3$ solves $x - 2 = 2x + 1$ perfectly well, but it doesn't solve the modulus equation.

For inequalities, find where the graphs meet, then read off from the sketch where one is above the other.

MODULUS INEQUALITIES

- $|x - a| < b$ means $a - b < x < a + b$: x is within b of a .
- $|x - a| > b$ means $x < a - b$ or $x > a + b$.
- For anything more complicated, sketch both sides and use the points where they meet.

WORKED EXAMPLE 4.4 · AN INEQUALITY FROM THE GRAPH

Solve $|2x - 3| > x + 1$.

The graphs meet at $x = \frac{2}{3}$ and $x = 4$ (Example 4.2). The V is above the line outside those points, so

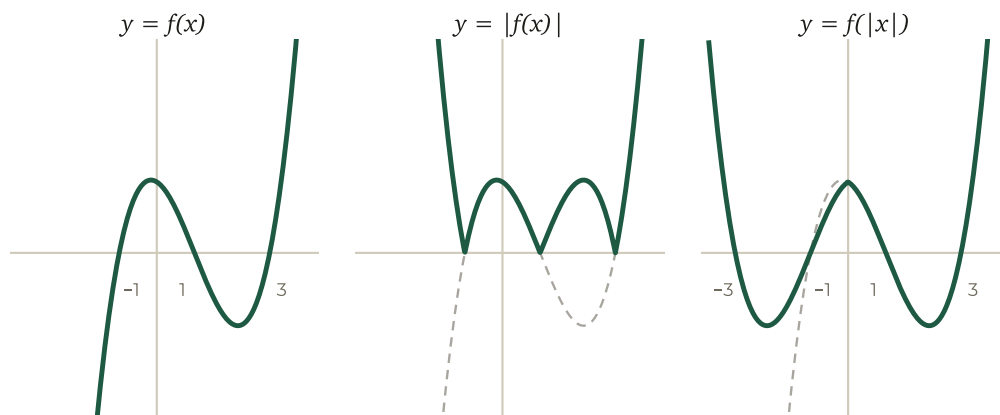
$$x < \frac{2}{3} \quad \text{or} \quad x > 4$$

3. The graphs of $|f(x)|$ and $f(|x|)$

The modulus can go on the outside of a function or on the inside, and the two give different graphs.

KEY RESULT

- $y = |f(x)|$: any part of $y = f(x)$ **below the x -axis** is reflected in the x -axis. Nothing is ever below the axis.
- $y = f(|x|)$: the part of $y = f(x)$ for $x \geq 0$ is kept, and **reflected in the y -axis** to replace the left half. The graph is symmetrical about the y -axis.



Left to right: $y = f(x)$, $y = |f(x)|$ and $y = f(|x|)$, for $f(x) = (x + 1)(x - 1)(x - 3)$.

COMMON MISTAKE

The two are easy to swap. Ask what the modulus acts on. In $|f(x)|$ it acts on the **output**, the y -value, so it folds the graph up over the x -axis. In $f(|x|)$ it acts on the **input**, the x -value, so it mirrors across the y -axis.

WORKED EXAMPLE 4.5 · FOUR SOLUTIONS

Solve $|x^2 - 4x| = 3$.

1. The graph of $y = |x^2 - 4x|$ is the parabola with its dip between $x = 0$ and $x = 4$ folded up, so its bump has height 4 at $x = 2$. The line $y = 3$ lies below that height, so it meets the graph four times.
2. Case $x^2 - 4x = 3$: $x^2 - 4x - 3 = 0$, so $x = 2 \pm \sqrt{7}$.
3. Case $x^2 - 4x = -3$: $x^2 - 4x + 3 = 0$, so $x = 1$ or $x = 3$.
4. The right-hand side is positive, so all four stand: $x = 2 - \sqrt{7}$, 1, 3, $2 + \sqrt{7}$.

4. Combining transformations

In Year 1 you met the four basic transformations one at a time. A function such as $y = 3f(2x) + 1$ applies several at once.

COMBINING TRANSFORMATIONS

- **Outside** the bracket acts on y , in the order you would calculate: $y = af(x) + d$ is a vertical stretch by scale factor a , **then** a translation up by d .
- **Inside** the bracket acts on x , in reverse: $y = f(bx + c)$ is a translation left by c , **then** a horizontal stretch by scale factor $\frac{1}{b}$.
- A point (p, q) on $y = f(x)$ moves to $\left(\frac{p-c}{b}, aq + d\right)$ on $y = af(bx + c) + d$.

WHY IT WORKS

The graph of $y = af(bx + c) + d$ passes through (x, y) when the original function passes through $(bx + c, \frac{y-d}{a})$. So a point (p, q) on $y = f(x)$ reappears where $bx + c = p$ and $y = aq + d$. Undoing “multiply by b , then add c ” means subtracting c first, then dividing by b , which is why the inside changes run in reverse.

WORKED EXAMPLE 4.6 · WHERE A POINT MOVES TO

The curve $y = f(x)$ has a minimum at $(4, -2)$. Find the minimum of $y = 3f(2x) + 1$.

1. Inside: $2x = 4$, so $x = 2$.
2. Outside: $y = 3(-2) + 1 = -5$.
3. A vertical stretch by a positive factor keeps a minimum a minimum, so the minimum is at $(2, -5)$.

WORKED EXAMPLE 4.7 · A TRANSFORMED V

Sketch $y = 2|x - 1| - 3$, giving the vertex and the points where it meets the axes.

1. Start from $y = |x|$: translate right by 1, stretch vertically by 2, then translate down by 3. The vertex moves from $(0, 0)$ to $(1, -3)$.
2. x -axis: $2|x - 1| = 3$, so $|x - 1| = \frac{3}{2}$ and $x = -\frac{1}{2}$ or $x = \frac{5}{2}$.
3. y -axis: $y = 2|-1| - 3 = -1$, so it meets it at $(0, -1)$.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[2 marks]

Find the coordinates of the points where $y = |3x + 6|$ meets the coordinate axes.

SOLUTION

The vertex is where $3x + 6 = 0$, so it meets the x -axis at $(-2, 0)$. When $x = 0$, $y = |6| = 6$, so it meets the y -axis at $(0, 6)$.

Question 2

[3 marks]

Solve:

(a) $|x - 4| = 3$

(b) $|2x + 1| = 5$

SOLUTION

(a) $x - 4 = 3$ or $x - 4 = -3$, so $x = 7$ or $x = 1$.

(b) $2x + 1 = 5$ or $2x + 1 = -5$, so $x = 2$ or $x = -3$.

Question 3

[2 marks]

Solve $|x + 1| < 4$.

SOLUTION

$|x + 1| = |x - (-1)|$ is the distance from x to -1 , and it must be less than 4:

$$-1 - 4 < x < -1 + 4$$

$$-5 < x < 3$$

Question 4

[3 marks]

The point $P(3, 5)$ lies on the curve $y = f(x)$. Find the image of P on each curve.

(a) $y = f(x + 2)$

(b) $y = 2f(x) - 1$

(c) $y = f(3x)$

SOLUTION

(a) A translation left by 2: $(1, 5)$.

(b) A vertical stretch by 2, then down by 1: $(3, 2 \times 5 - 1) = (3, 9)$.

(c) $3x = 3$ gives $x = 1$: $(1, 5)$.

Standard**Question 5**

[4 marks]

Solve $|3x - 2| = x + 4$.

SOLUTION

First case, $3x - 2 = x + 4$: $x = 3$.

Second case, $-(3x - 2) = x + 4$, so $2 - 3x = x + 4$ and $x = -\frac{1}{2}$.

Check: at $x = 3$ both sides are 7; at $x = -\frac{1}{2}$ both sides are $\frac{7}{2}$. So $x = 3$ or $x = -\frac{1}{2}$.

Question 6

[4 marks]

Solve $|x + 3| = 2x$.

SOLUTION

First case, $x + 3 = 2x$: $x = 3$. Check: $|6| = 6 = 2(3)$. ✓

Second case, $-(x + 3) = 2x$: $x = -1$. But then $2x = -2$, which is negative, so reject it.

The only solution is $x = 3$.

Question 7

[4 marks]

Solve $|2x - 1| > |x + 2|$.

SOLUTION

Both sides are non-negative, so squaring keeps the inequality the same way round:

$$\begin{aligned}(2x - 1)^2 &> (x + 2)^2 \\ 4x^2 - 4x + 1 &> x^2 + 4x + 4 \\ 3x^2 - 8x - 3 &> 0 \\ (3x + 1)(x - 3) &> 0\end{aligned}$$

A quadratic with positive x^2 coefficient is positive outside its roots, so $x < -\frac{1}{3}$ or $x > 3$.

Question 8

[4 marks]

$$f(x) = x^2 - 2x - 3.$$

- (a) Sketch $y = |f(x)|$, giving the coordinates of the points where it meets the axes and of the maximum between them. [3]
- (b) State the values of k for which $|f(x)| = k$ has exactly four solutions. [1]

SOLUTION

(a) $f(x) = (x + 1)(x - 3)$, so the graph meets the x -axis at $(-1, 0)$ and $(3, 0)$. Between them $f(x)$ is negative, with minimum $f(1) = -4$; the modulus folds this up to a maximum at $(1, 4)$. It meets the y -axis at $(0, |-3|) = (0, 3)$.

(b) A horizontal line $y = k$ meets the graph four times when it cuts the folded bump and both outer arms: $0 < k < 4$.

Question 9

[3 marks]

$$f(x) = x^2 - 5x + 4.$$

- (a) Write down an expression for $f(|x|)$ when $x < 0$. [1]
- (b) Solve $f(|x|) = 0$. [2]

SOLUTION

(a) For $x < 0$, $|x| = -x$, so $f(|x|) = (-x)^2 - 5(-x) + 4 = x^2 + 5x + 4$.

(b) $f(|x|) = (|x| - 1)(|x| - 4) = 0$, so $|x| = 1$ or $|x| = 4$. That gives $x = \pm 1$ or $x = \pm 4$.

Exam style**Question 10**

[6 marks]

$$f(x) = 2|x - 1| - 3.$$

(a) Sketch $y = f(x)$, giving the coordinates of the vertex and of the points where the graph meets the axes. [3]

(b) Solve $f(x) = -x$. [3]

SOLUTION

(a) The vertex is at $(1, -3)$. It meets the x -axis where $|x - 1| = \frac{3}{2}$, at $(-\frac{1}{2}, 0)$ and $(\frac{5}{2}, 0)$, and the y -axis at $(0, -1)$.

(b) For $x \geq 1$: $2(x - 1) - 3 = -x$, so $3x = 5$ and $x = \frac{5}{3}$, which is at least 1. ✓

For $x < 1$: $-2(x - 1) - 3 = -x$, so $-2x - 1 = -x$ and $x = -1$, which is less than 1. ✓

So $x = \frac{5}{3}$ or $x = -1$.

Question 11

[6 marks]

The curve $y = f(x)$ has a maximum at $(2, 5)$ and meets the y -axis at $(0, 1)$. For each curve below, find the images of these two points, and say whether the image of the maximum is a maximum or a minimum.

(a) $y = f(x - 3) + 2$ [2]

(b) $y = -2f(x)$ [2]

(c) $y = f(2x + 4)$ [2]

SOLUTION

(a) Right 3, up 2: the maximum moves to $(5, 7)$ and is still a maximum; $(0, 1)$ moves to $(3, 3)$.

(b) Each y -value is multiplied by -2 : $(2, -10)$, which is now a **minimum**, and $(0, -2)$.

(c) Solve $2x + 4 = 2$: $x = -1$, so the maximum is at $(-1, 5)$ and is still a maximum. Solve $2x + 4 = 0$: $x = -2$, so $(0, 1)$ moves to $(-2, 1)$.

Question 12

[6 marks]

- (a) Sketch $y = |x^2 - 4|$, giving the coordinates of the points where it meets the axes. [2]
- (b) Solve $|x^2 - 4| = 3x$. [4]

SOLUTION

(a) $x^2 - 4 = 0$ at $x = \pm 2$, so the graph meets the x -axis at $(-2, 0)$ and $(2, 0)$. Between them the parabola dips to -4 , which folds up to a maximum at $(0, 4)$, where it meets the y -axis.

(b) Any solution needs $3x \geq 0$, so $x \geq 0$.

Case $x^2 - 4 = 3x$: $x^2 - 3x - 4 = 0$, so $(x - 4)(x + 1) = 0$. Only $x = 4$ has $x \geq 0$.

Case $-(x^2 - 4) = 3x$: $x^2 + 3x - 4 = 0$, so $(x + 4)(x - 1) = 0$. Only $x = 1$ has $x \geq 0$.

Check: $|16 - 4| = 12 = 3(4)$ and $|1 - 4| = 3 = 3(1)$. So $x = 1$ or $x = 4$.