

YEAR 2 PURE · CHAPTER 15

Integration Techniques

Specification: Edexcel Pure 8.2, 8.5, 8.6 — Integration · AQA section H2, H5, H6

KEY FACTS — THIS CHAPTER

- $\int e^{kx} dx = \frac{1}{k}e^{kx} + c$, $\int \frac{1}{x} dx = \ln|x| + c$, $\int \cos kx dx = \frac{1}{k} \sin kx + c$ and $\int \sin kx dx = -\frac{1}{k} \cos kx + c$.
- **Reverse chain rule:** $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$, and $\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + c$.
- Use identities to turn $\sin^2 x$, $\cos^2 x$ and $\tan^2 x$ into things you can integrate.
- **Substitution:** replace x by u , dx by $\frac{dx}{du} du$, and change the limits.
- **By parts:** $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$. Choose u to be the factor that gets simpler when differentiated, except that $\ln x$ is always u .
- **Partial fractions** split a fraction into pieces that each integrate to a logarithm.

Year 1 integrated powers of x . This chapter reverses every derivative from Chapter 12, then adds methods for the integrals that don't match anything directly: spotting a reversed chain rule, substituting a new variable, integrating a product by parts, and splitting a fraction.

1. Standard integrals

Each derivative from Chapter 12, read backwards, gives an integral.

KEY RESULT

$$\int e^{kx} dx = \frac{1}{k} e^{kx} + c$$

$$\int \frac{1}{x} dx = \ln |x| + c$$

$$\int \cos kx dx = \frac{1}{k} \sin kx + c$$

$$\int \sin kx dx = -\frac{1}{k} \cos kx + c$$

$$\int \sec^2 kx dx = \frac{1}{k} \tan kx + c$$

More generally, if $\int f(x) dx = F(x) + c$, then $\int f(ax + b) dx = \frac{1}{a} F(ax + b) + c$.

WORKED EXAMPLE 15.1 · A SUM OF STANDARD INTEGRALS

Find $\int \left(e^{3x} + \frac{4}{x} - 2 \sin 5x \right) dx$.

Integrate term by term:

$$\frac{1}{3} e^{3x} + 4 \ln |x| + \frac{2}{5} \cos 5x + c$$

WORKED EXAMPLE 15.2 · A DEFINITE INTEGRAL

Find the exact value of $\int_0^{\pi/6} \sec^2 2x dx$.

$$\begin{aligned} \int_0^{\pi/6} \sec^2 2x dx &= \left[\frac{1}{2} \tan 2x \right]_0^{\pi/6} \\ &= \frac{1}{2} \tan \frac{\pi}{3} - 0 = \frac{\sqrt{3}}{2} \end{aligned}$$

COMMON MISTAKE

The integral of $\frac{1}{x}$ is $\ln|x|$, with the modulus, because $\ln x$ is only defined for $x > 0$. So $\int_{-3}^{-1} \frac{1}{x} dx = \ln 1 - \ln 3 = -\ln 3$, which is negative, as it should be for an area below the axis.

2. Recognising a reversed chain rule

Differentiating $\ln|f(x)|$ or $(f(x))^{n+1}$ by the chain rule gives a recognisable pattern: something multiplied by the derivative of its inside. When an integral has that pattern, you can reverse the chain rule directly.

KEY RESULT

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + c$$

The second works for any $n \neq -1$. If the top is only a multiple of $f'(x)$, adjust by that constant.

WORKED EXAMPLE 15.3 · THE TOP IS THE DERIVATIVE OF THE BOTTOM

Find (a) $\int \frac{2x}{x^2+1} dx$ and (b) $\int \tan x dx$.

(a) The top is the derivative of $x^2 + 1$, so the integral is $\ln(x^2 + 1) + c$. No modulus is needed, because $x^2 + 1 > 0$.

(b) $\tan x = \frac{\sin x}{\cos x}$, and $\sin x$ is minus the derivative of $\cos x$. So

$$\begin{aligned} \int \tan x dx &= -\ln|\cos x| + c \\ &= \ln|\sec x| + c \end{aligned}$$

WORKED EXAMPLE 15.4 · A POWER OF A BRACKET

Find $\int x(x^2 + 3)^4 dx$.

The derivative of $x^2 + 3$ is $2x$, and the integral has x , which is half of it. So try $(x^2 + 3)^5$: its derivative is $10x(x^2 + 3)^4$, which is ten times too big. Adjust:

$$\int x(x^2 + 3)^4 dx = \frac{1}{10}(x^2 + 3)^5 + c$$

Using trigonometric identities

$\sin^2 x$ and $\cos^2 x$ have no standard integral, but the double-angle formulae of Chapter 10 turn them into ones that do. $\tan^2 x$ becomes $\sec^2 x - 1$.

KEY RESULT

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\tan^2 x = \sec^2 x - 1$$

WORKED EXAMPLE 15.5 · SQUARES OF TRIG FUNCTIONS

Find $\int \sin^2 x dx$, and the exact value of $\int_0^{\pi/4} \tan^2 x dx$.

$$1. \int \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c.$$

2. For the second:

$$\begin{aligned} \int_0^{\pi/4} (\sec^2 x - 1) dx \\ = \left[\tan x - x \right]_0^{\pi/4} = 1 - \frac{\pi}{4} \end{aligned}$$

3. Integration by substitution

When the pattern isn't obvious, change the variable. Choose u as some function of x , then rewrite **everything** in terms of u , including dx .

KEY RESULT

To substitute $u = g(x)$:

1. Find $\frac{du}{dx}$ and use it to replace dx with $\frac{dx}{du} du$.
2. Write the rest of the integrand in terms of u .
3. For a definite integral, change the limits to values of u .
4. Integrate, and for an indefinite integral, put x back at the end.

WORKED EXAMPLE 15.6 · AN INDEFINITE SUBSTITUTION

Use the substitution $u = 2x + 1$ to find $\int x\sqrt{2x+1} dx$.

1. $\frac{du}{dx} = 2$, so $dx = \frac{1}{2} du$, and $x = \frac{u-1}{2}$.
2. Substitute and integrate:

$$\begin{aligned} & \int \frac{u-1}{2} \sqrt{u} \times \frac{1}{2} du \\ &= \frac{1}{4} \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) du \\ &= \frac{1}{10} u^{\frac{5}{2}} - \frac{1}{6} u^{\frac{3}{2}} + c \end{aligned}$$

3. Put x back: $\frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + c$.

WORKED EXAMPLE 15.7 · CHANGING THE LIMITS

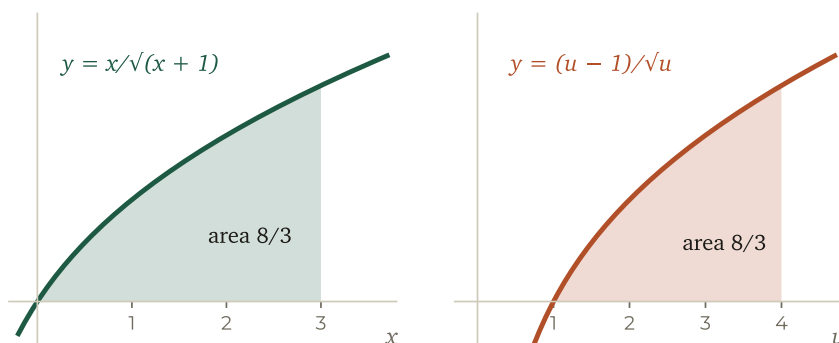
Use the substitution $u = x + 1$ to find $\int_0^3 \frac{x}{\sqrt{x+1}} dx$.

1. $dx = du$ and $x = u - 1$. When $x = 0$, $u = 1$, and when $x = 3$, $u = 4$.

2. So the integral becomes

$$\begin{aligned}
 & \int_1^4 \frac{u-1}{\sqrt{u}} du \\
 &= \int_1^4 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) du \\
 &= \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_1^4 \\
 &= \left(\frac{16}{3} - 4 \right) - \left(\frac{2}{3} - 2 \right) = \frac{8}{3}
 \end{aligned}$$

With the limits changed, there's no need to go back to x .



The substitution $u = x + 1$ turns the area on the left into the one on the right. They are equal: both are $8/3$.

4. Integration by parts

Integrating the product rule gives a way to integrate some products.

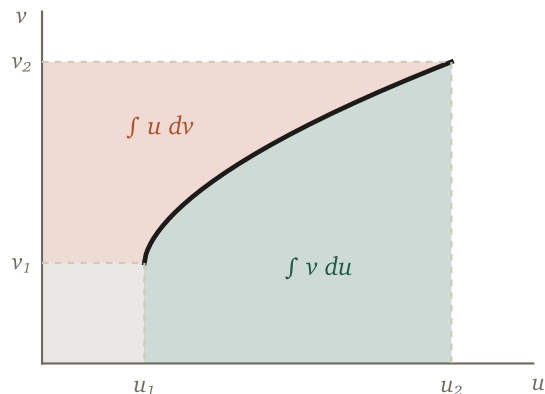
KEY RESULT

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Choose u to be the factor that gets simpler when you differentiate it, such as a power of x , so the new integral is easier. The exception is $\ln x$: it has no standard integral, so always make it u .

WHERE IT COMES FROM

The product rule says $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$. Integrate both sides with respect to x : $uv = \int u\frac{dv}{dx} dx + \int v\frac{du}{dx} dx$. Rearranging gives the formula. The figure shows the same thing with areas.



The two integrals are the two areas either side of the curve. Together they fill the difference between the rectangles,

$$u_2v_2 - u_1v_1.$$

WORKED EXAMPLE 15.8 · A POWER TIMES AN EXPONENTIAL

Find $\int xe^{2x} dx$.

1. Let $u = x$, which gets simpler, and $\frac{dv}{dx} = e^{2x}$. Then $\frac{du}{dx} = 1$ and $v = \frac{1}{2}e^{2x}$.
2. By parts:

$$\begin{aligned} \int xe^{2x} dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c \end{aligned}$$

WORKED EXAMPLE 15.9 · INTEGRATING LN X

Find $\int \ln x dx$, and hence the exact value of $\int_1^3 \ln x dx$.

1. Write $\ln x$ as $\ln x \times 1$. Let $u = \ln x$ and $\frac{dv}{dx} = 1$, so $\frac{du}{dx} = \frac{1}{x}$ and $v = x$:

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \times \frac{1}{x} \, dx \\ &= x \ln x - x + c\end{aligned}$$

2. So $\int_1^3 \ln x \, dx = (3 \ln 3 - 3) - (0 - 1) = 3 \ln 3 - 2$. This is the exact value the trapezium rule estimated in Chapter 14.

COMMON MISTAKE

Some integrals need parts twice. For $\int x^2 e^x \, dx$, the first use leaves $\int 2x e^x \, dx$, which needs parts again with $u = 2x$. Keep choosing the power of x as u each time, or the integral gets worse instead of better.

5. Using partial fractions

A fraction whose bottom factorises can be split into partial fractions (Chapter 2). Each piece with a linear bottom integrates to a logarithm.

KEY RESULT

$$\int \frac{A}{ax + b} \, dx = \frac{A}{a} \ln |ax + b| + c$$

WORKED EXAMPLE 15.10 · A DEFINITE INTEGRAL WITH PARTIAL FRACTIONS

Find the exact value of $\int_2^3 \frac{1}{x^2 - 1} \, dx$.

1. $x^2 - 1 = (x - 1)(x + 1)$, and the partial fractions are

$$\frac{1}{x^2 - 1} = \frac{1}{2} \left(\frac{1}{x - 1} - \frac{1}{x + 1} \right)$$

2. Integrate and use the laws of logarithms:

$$\begin{aligned} & \frac{1}{2} \left[\ln |x-1| - \ln |x+1| \right]_2^3 \\ &= \frac{1}{2} [(\ln 2 - \ln 4) - (\ln 1 - \ln 3)] \\ &= \frac{1}{2} \ln \frac{3}{2} \end{aligned}$$

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[4 marks]

Find each of these.

(a) $\int e^{4x} dx$ [1]

(b) $\int \frac{3}{x} dx$ [1]

(c) $\int \cos 2x dx$ [1]

(d) $\int \sec^2 3x dx$ [1]

SOLUTION

(a) $\frac{1}{4}e^{4x} + c$ (b) $3 \ln |x| + c$ (c) $\frac{1}{2} \sin 2x + c$ (d) $\frac{1}{3} \tan 3x + c$

Question 2

[2 marks]

Find the exact value of $\int_0^{\pi/2} \sin 2x dx$.

SOLUTION

$$\left[-\frac{1}{2} \cos 2x \right]_0^{\pi/2} = \frac{1}{2} - \left(-\frac{1}{2} \right) = 1$$

Question 3

[2 marks]

Find $\int \frac{6x^2}{x^3 + 1} dx$.

SOLUTION

The top is twice the derivative of $x^3 + 1$, so the integral is $2 \ln |x^3 + 1| + c$.

Question 4

[2 marks]

Find $\int \cos^2 x dx$.

SOLUTION

$$\begin{aligned} \int \frac{1}{2}(1 + \cos 2x) dx \\ = \frac{1}{2}x + \frac{1}{4} \sin 2x + c \end{aligned}$$

Standard**Question 5**

[3 marks]

Find $\int x \cos x dx$.

SOLUTION

By parts with $u = x$ and $\frac{dv}{dx} = \cos x$, so $v = \sin x$:

$$\begin{aligned} \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + c \end{aligned}$$

Question 6

[4 marks]

Use the substitution $u = x - 2$ to find $\int x(x - 2)^6 dx$.

SOLUTION

$dx = du$ and $x = u + 2$, so

$$\begin{aligned}\int (u + 2)u^6 du &= \int (u^7 + 2u^6) du \\ &= \frac{1}{8}u^8 + \frac{2}{7}u^7 + c \\ &= \frac{1}{8}(x - 2)^8 + \frac{2}{7}(x - 2)^7 + c\end{aligned}$$

Question 7

[5 marks]

Find the exact value of $\int_1^e x \ln x dx$.

SOLUTION

$\ln x$ is u , and $\frac{dv}{dx} = x$, so $v = \frac{1}{2}x^2$:

$$\begin{aligned}\int x \ln x dx &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c\end{aligned}$$

So the definite integral is

$$\left(\frac{1}{2}e^2 - \frac{1}{4}e^2\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4}(e^2 + 1)$$

Question 8

[5 marks]

(a) Express $\frac{x + 5}{(x + 1)(x - 3)}$ in partial fractions. [3]

(b) Hence find $\int \frac{x + 5}{(x + 1)(x - 3)} dx$. [2]

SOLUTION

(a) $x + 5 \equiv A(x - 3) + B(x + 1)$. Put $x = 3$: $8 = 4B$, so $B = 2$. Put $x = -1$: $4 = -4A$, so $A = -1$. So the fraction is $\frac{2}{x - 3} - \frac{1}{x + 1}$.

(b) $2 \ln |x - 3| - \ln |x + 1| + c$.

Question 9

[3 marks]

Find the exact value of $\int_0^{\pi/4} \sin x \cos x \, dx$.

SOLUTION

$\sin x \cos x = \frac{1}{2} \sin 2x$, so

$$\begin{aligned} & \int_0^{\pi/4} \frac{1}{2} \sin 2x \, dx \\ &= \left[-\frac{1}{4} \cos 2x \right]_0^{\pi/4} = 0 + \frac{1}{4} = \frac{1}{4} \end{aligned}$$

Exam style**Question 10**

[6 marks]

Use the substitution $u = 2x - 1$ to find the exact value of $\int_1^5 \frac{x}{\sqrt{2x-1}} \, dx$.

SOLUTION

$\frac{du}{dx} = 2$, so $dx = \frac{1}{2} du$, and $x = \frac{u+1}{2}$. When $x = 1$, $u = 1$, and when $x = 5$, $u = 9$. So the integral is

$$\begin{aligned} & \int_1^9 \frac{u+1}{2\sqrt{u}} \times \frac{1}{2} du \\ &= \frac{1}{4} \int_1^9 \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}} \right) du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \right]_1^9 \\ &= \frac{1}{4} \left[(18 + 6) - \left(\frac{2}{3} + 2 \right) \right] = \frac{16}{3} \end{aligned}$$

Question 11

[6 marks]

(a) Find $\int x^2 e^x \, dx$. [4]

(b) Hence find the exact value of $\int_0^1 x^2 e^x \, dx$. [2]

SOLUTION

(a) By parts with $u = x^2$ and $v = e^x$, then by parts again on the new integral, with $u = 2x$:

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - \int 2x e^x dx \\ &= x^2 e^x - \left(2x e^x - \int 2e^x dx \right) \\ &= e^x (x^2 - 2x + 2) + c\end{aligned}$$

(b) $e(1 - 2 + 2) - e^0(0 - 0 + 2) = e - 2.$

Question 12

[7 marks]

(a) Express $\frac{3}{(x+1)(x-2)}$ in partial fractions. [3]

(b) Hence show that $\int_3^5 \frac{3}{(x+1)(x-2)} dx = \ln 2.$ [4]

SOLUTION

(a) $3 \equiv A(x-2) + B(x+1)$. Put $x = 2$: $3 = 3B$, so $B = 1$. Put $x = -1$: $3 = -3A$, so $A = -1$. The fraction is $\frac{1}{x-2} - \frac{1}{x+1}$.

(b)

$$\begin{aligned}&\left[\ln|x-2| - \ln|x+1| \right]_3^5 \\ &= (\ln 3 - \ln 6) - (\ln 1 - \ln 4) \\ &= \ln \frac{3 \times 4}{6} = \ln 2\end{aligned}$$