

YEAR 2 PURE · CHAPTER 13

Implicit and Parametric Differentiation

Specification: Edexcel Pure 7.1, 7.3–7.5 — Differentiation · AQA section G1, G3, G4, G5

KEY FACTS — THIS CHAPTER

- **Parametric:** if x and y are functions of t , then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
- **Implicit:** differentiate every term with respect to x . A term in y gets an extra factor $\frac{dy}{dx}$ by the chain rule, so $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$.
- A product such as xy needs the product rule: $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$.
- **Connected rates:** link the rate you want to the one you know with the chain rule, such as $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$.
- A curve is **convex** where $f''(x) \geq 0$ and **concave** where $f''(x) \leq 0$. A **point of inflection** is where it changes from one to the other, so $f''(x)$ changes sign.

Not every curve comes as $y = f(x)$. Some are parametric, some are given by an equation like $x^2 + y^2 = 25$ that mixes x and y , and some quantities change with time together. The chain rule handles all three. The last section uses the second derivative to describe which way a curve bends.

1. Parametric differentiation

KEY RESULT

If x and y are both functions of a parameter t , then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy/dt}{dx/dt}$$

wherever $\frac{dx}{dt} \neq 0$. The answer is in terms of t .

WORKED EXAMPLE 13.1 · A TANGENT TO A PARAMETRIC CURVE

A curve has parametric equations $x = t^2$, $y = t^3 - 3t$. Find the equation of the tangent at the point where $t = 2$.

1. Differentiate each with respect to t and divide:

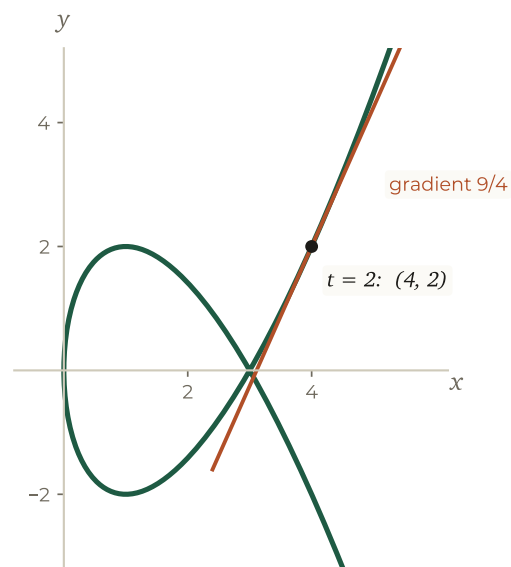
$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

2. At $t = 2$ the gradient is $\frac{9}{4}$, and the point is $(4, 2)$.

3. The tangent is

$$y - 2 = \frac{9}{4}(x - 4)$$

$$4y = 9x - 28$$



$x = t^2$, $y = t^3 - 3t$, with its tangent at $t = 2$.

WORKED EXAMPLE 13.2 · A NORMAL TO AN ELLIPSE

A curve has parametric equations $x = 2 \cos \theta$, $y = 3 \sin \theta$. Find the equation of the normal at the point where $\theta = \frac{\pi}{4}$.

1. $\frac{dx}{d\theta} = -2 \sin \theta$ and $\frac{dy}{d\theta} = 3 \cos \theta$, so

$$\frac{dy}{dx} = \frac{3 \cos \theta}{-2 \sin \theta} = -\frac{3}{2} \cot \theta$$

2. At $\theta = \frac{\pi}{4}$ the tangent's gradient is $-\frac{3}{2}$, so the normal's is $\frac{2}{3}$. The point is $(\sqrt{2}, \frac{3\sqrt{2}}{2})$.

3. The normal is $y - \frac{3\sqrt{2}}{2} = \frac{2}{3}(x - \sqrt{2})$. Multiplying by 6 and rearranging gives $6y = 4x + 5\sqrt{2}$.

2. Implicit differentiation

An equation like $x^2 + y^2 = 25$ defines y in terms of x without solving for it: the relationship is **implicit**. You can still differentiate every term with respect to x .

KEY RESULT

To differentiate a function of y with respect to x , differentiate it with respect to y , then multiply by $\frac{dy}{dx}$ (the chain rule):

$$\frac{d}{dx}(f(y)) = f'(y) \frac{dy}{dx}$$

Terms that mix x and y , such as xy , need the product rule as well.

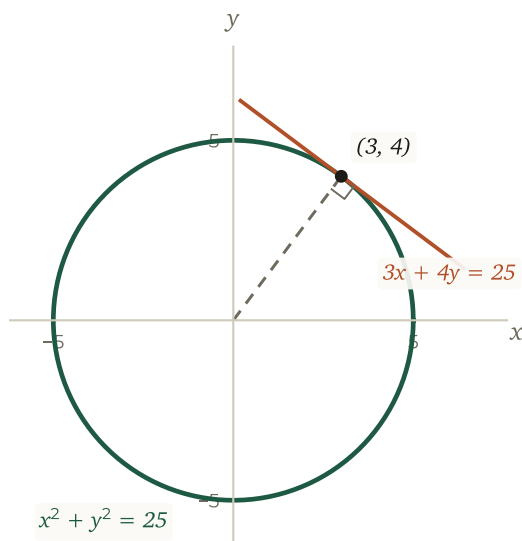
WORKED EXAMPLE 13.3 · A TANGENT TO A CIRCLE

Find the equation of the tangent to $x^2 + y^2 = 25$ at the point $(3, 4)$.

1. Differentiate every term with respect to x :

$$\begin{aligned} 2x + 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

2. At $(3, 4)$ the gradient is $-\frac{3}{4}$. The tangent is $y - 4 = -\frac{3}{4}(x - 3)$, which is $3x + 4y = 25$.



The tangent at $(3, 4)$ has gradient $-x/y = -3/4$: at right angles to the radius, as it should be.

WORKED EXAMPLE 13.4 · A PRODUCT TERM

A curve has equation $x^3 + xy + y^3 = 11$. Find the gradient of the curve at the point $(1, 2)$.

1. Differentiate term by term, using the product rule on xy :

$$3x^2 + x \frac{dy}{dx} + y + 3y^2 \frac{dy}{dx} = 0$$

2. Collect the $\frac{dy}{dx}$ terms:

$$\frac{dy}{dx} = -\frac{3x^2 + y}{x + 3y^2}$$

3. At $(1, 2)$ the gradient is $-\frac{3+2}{1+12} = -\frac{5}{13}$.

WORKED EXAMPLE 13.5 · STATIONARY POINTS OF AN IMPLICIT CURVE

Find the stationary points of the curve $x^2 + xy + y^2 = 12$.

1. Differentiate and rearrange:

$$2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

2. The gradient is 0 when $2x + y = 0$, that is $y = -2x$. Substitute into the curve's equation:

$$x^2 - 2x^2 + 4x^2 = 12 \Rightarrow x = \pm 2$$

3. The stationary points are $(2, -4)$ and $(-2, 4)$.

ANOTHER ROUTE TO THE EXPONENTIAL DERIVATIVE

If $y = a^x$, then $\ln y = x \ln a$. Differentiating implicitly, $\frac{1}{y} \frac{dy}{dx} = \ln a$, so $\frac{dy}{dx} = y \ln a = a^x \ln a$, as in Chapter 12.

3. Connected rates of change

When two quantities are linked, their rates of change are linked too. The chain rule connects them.

KEY RESULT

If V is a function of r , and r changes with time t ,

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Write down the rate you know and the rate you want. Then find the link between the variables and differentiate it.

WORKED EXAMPLE 13.6 · A BALLOON

A spherical balloon is inflated at a rate of 50 cm^3 per second. Find the rate at which its radius is increasing when the radius is 10 cm .

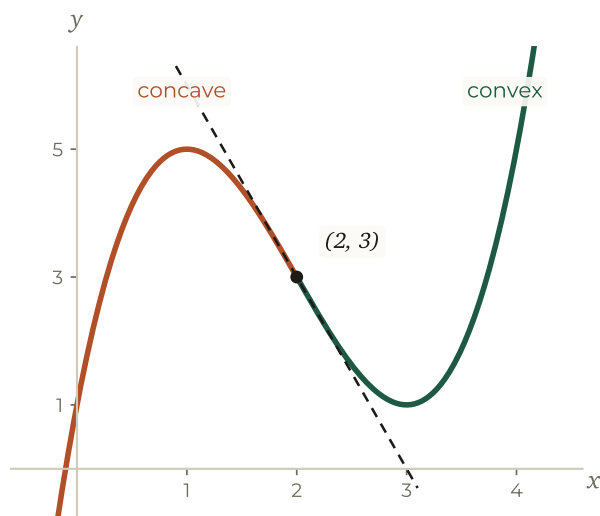
1. You know $\frac{dV}{dt} = 50$ and want $\frac{dr}{dt}$. The link is $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$, which is 400π when $r = 10$.
2. So

$$\frac{dr}{dt} = \frac{dV/dt}{dV/dr} = \frac{50}{400\pi} = \frac{1}{8\pi}$$

That is about 0.0398 cm per second.

4. Concave and convex curves**KEY RESULT**

- A curve is **convex** where $f''(x) \geq 0$: its gradient is increasing, and it bends upwards like a bowl.
- A curve is **concave** where $f''(x) \leq 0$: its gradient is decreasing, and it bends downwards like an arch.
- A **point of inflection** is where the curve changes from concave to convex, or the other way. There $f''(x) = 0$ and **changes sign**.



$y = x^3 - 6x^2 + 9x + 1$ is concave for $x < 2$ and convex for $x > 2$. At the point of inflection the tangent crosses the curve.

WORKED EXAMPLE 13.7 · FINDING A POINT OF INFLECTION

For $f(x) = x^3 - 6x^2 + 9x + 1$, find where the curve is concave, where it is convex, and its point of inflection.

1. $f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$.
2. $f''(x) \leq 0$ for $x \leq 2$, so the curve is concave there. $f''(x) \geq 0$ for $x \geq 2$, so it is convex there.
3. $f''(x)$ changes sign at $x = 2$, and $f(2) = 8 - 24 + 18 + 1 = 3$. So $(2, 3)$ is a point of inflection.

WORKED EXAMPLE 13.8 · CONVEX EVERYWHERE

Show that $f(x) = x \ln x$ is convex for all $x > 0$.

By the product rule, $f'(x) = \ln x + 1$, so $f''(x) = \frac{1}{x}$. This is positive for every $x > 0$, so the curve is convex there.

COMMON MISTAKE

$f''(x) = 0$ is not enough on its own. For $y = x^4$, $f''(x) = 12x^2$ is 0 at $x = 0$, but it is positive on both sides, so the curve is convex throughout and has no point of inflection. Always check that $f''(x)$ changes sign.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[2 marks]

A curve has parametric equations $x = 3t^2$, $y = 2t^3$. Find $\frac{dy}{dx}$ in terms of t .

SOLUTION

$$\frac{dy}{dx} = \frac{6t^2}{6t} = t$$

Question 2

[2 marks]

Find $\frac{dy}{dx}$ for the curve $y^2 = 4x$, in terms of y .

SOLUTION

Differentiating implicitly, $2y \frac{dy}{dx} = 4$, so $\frac{dy}{dx} = \frac{2}{y}$.

Question 3

[2 marks]

Find $\frac{dy}{dx}$ for the curve $x^2y = 8$, in terms of x and y .

SOLUTION

By the product rule, $2xy + x^2 \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2y}{x}$.

Question 4

[2 marks]

Find the values of x for which the curve $y = x^3 - 3x$ is concave.

SOLUTION

$\frac{d^2y}{dx^2} = 6x$, which is ≤ 0 when $x \leq 0$. So the curve is concave for $x \leq 0$.

Standard**Question 5**

[5 marks]

A curve has parametric equations $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t > 0$.

(a) Find $\frac{dy}{dx}$ in terms of t . [2]

(b) Find the equation of the tangent at the point where $t = 2$. [3]

SOLUTION

(a) $\frac{dx}{dt} = 1 - \frac{1}{t^2}$ and $\frac{dy}{dt} = 1 + \frac{1}{t^2}$. Multiplying top and bottom by t^2 :

$$\frac{dy}{dx} = \frac{t^2 + 1}{t^2 - 1}$$

(b) At $t = 2$ the gradient is $\frac{5}{3}$ and the point is $(\frac{5}{2}, \frac{3}{2})$. The tangent is $y - \frac{3}{2} = \frac{5}{3}(x - \frac{5}{2})$, which is $3y = 5x - 8$.

Question 6

[5 marks]

A curve has equation $x^2 + 3xy - y^2 = 3$.

(a) Find $\frac{dy}{dx}$ in terms of x and y . [3]

(b) Find the equation of the normal at the point $(1, 1)$. [2]

SOLUTION

(a) Differentiating implicitly, with the product rule on $3xy$:

$$2x + 3x\frac{dy}{dx} + 3y - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x + 3y}{3x - 2y}$$

(b) At $(1, 1)$ the gradient is -5 , so the normal has gradient $\frac{1}{5}$: $y - 1 = \frac{1}{5}(x - 1)$, which is $5y = x + 4$.

Question 7

[3 marks]

The edges of a cube are increasing at 0.1 cm per second. Find the rate at which its volume is increasing when the edges are 5 cm long.

SOLUTION

$V = x^3$, so $\frac{dV}{dx} = 3x^2 = 75$ when $x = 5$. Then

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dx} \times \frac{dx}{dt} \\ &= 75 \times 0.1 = 7.5 \text{ cm}^3 \text{ per second}\end{aligned}$$

Question 8

[5 marks]

For $f(x) = x^4 - 6x^2 + 5$, find the values of x for which the curve $y = f(x)$ is concave, and the coordinates of its points of inflection.

SOLUTION

$$f'(x) = 4x^3 - 12x \text{ and } f''(x) = 12x^2 - 12 = 12(x-1)(x+1).$$

$f''(x) \leq 0$ for $-1 \leq x \leq 1$, so the curve is concave there, and convex for $|x| \geq 1$. $f''(x)$ changes sign at $x = \pm 1$, where $f(\pm 1) = 1 - 6 + 5 = 0$. So the points of inflection are $(1, 0)$ and $(-1, 0)$.

Question 9

[4 marks]

Show that the curve $y = xe^x$ has exactly one point of inflection, and find its coordinates.

SOLUTION

By the product rule, $\frac{dy}{dx} = e^x(x+1)$, and again, $\frac{d^2y}{dx^2} = e^x(x+2)$.

$e^x > 0$, so $\frac{d^2y}{dx^2}$ has the sign of $x+2$. It changes sign only at $x = -2$, so there is exactly one point of inflection, at $(-2, -2e^{-2})$.

Exam style**Question 10**

[7 marks]

A curve has parametric equations $x = \cos t$, $y = \sin 2t$, $0 \leq t \leq \pi$.

(a) Find $\frac{dy}{dx}$ in terms of t . [2]

(b) Find the coordinates of the points where the tangent is horizontal. [3]

- (c) Find the coordinates of the points where the tangent is vertical. [2]

SOLUTION

(a)

$$\frac{dy}{dx} = \frac{2 \cos 2t}{-\sin t} = -\frac{2 \cos 2t}{\sin t}$$

(b) Horizontal when $\cos 2t = 0$. With $0 \leq 2t \leq 2\pi$, $2t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, so $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. The points are $\left(\frac{\sqrt{2}}{2}, 1\right)$ and $\left(-\frac{\sqrt{2}}{2}, -1\right)$.

(c) Vertical when $\frac{dx}{dt} = -\sin t = 0$ (and $\frac{dy}{dt} \neq 0$), so $t = 0$ or π : the points $(1, 0)$ and $(-1, 0)$.

Question 11

[7 marks]

A curve has equation $2x^2 + y^2 - 2xy = 25$.

(a) Show that $\frac{dy}{dx} = \frac{y - 2x}{y - x}$. [4]

(b) Find the exact coordinates of the points where the tangent is parallel to the x -axis. [3]

SOLUTION

(a) Differentiating implicitly, with the product rule on $2xy$:

$$\begin{aligned} 4x + 2y \frac{dy}{dx} - 2x \frac{dy}{dx} - 2y &= 0 \\ (2y - 2x) \frac{dy}{dx} &= 2y - 4x \end{aligned}$$

Dividing by $2(y - x)$ gives $\frac{dy}{dx} = \frac{y - 2x}{y - x}$.

(b) The gradient is 0 when $y = 2x$. Substituting, $2x^2 + 4x^2 - 4x^2 = 25$, so $x^2 = \frac{25}{2}$ and $x = \pm \frac{5\sqrt{2}}{2}$. The points are $\left(\frac{5\sqrt{2}}{2}, 5\sqrt{2}\right)$ and $\left(-\frac{5\sqrt{2}}{2}, -5\sqrt{2}\right)$.

Question 12

[7 marks]

The area of a circular ripple increases at a constant 12 cm^2 per second.

- (a) Find the rate at which the radius is increasing when the radius is 3 cm. [3]
- (b) Find the rate at which the radius is increasing when the area is $100\pi \text{ cm}^2$. [2]
- (c) Find the rate at which the circumference is increasing when the radius is 3 cm. [2]

SOLUTION

(a) $A = \pi r^2$, so $\frac{dA}{dr} = 2\pi r$. Then

$$\frac{dr}{dt} = \frac{dA/dt}{dA/dr} = \frac{12}{2\pi r}$$

which is $\frac{12}{6\pi} = \frac{2}{\pi} = 0.637$ cm per second when $r = 3$.

(b) $\pi r^2 = 100\pi$ gives $r = 10$, so $\frac{dr}{dt} = \frac{12}{20\pi} = 0.191$ cm per second (3 s.f.).

(c) $C = 2\pi r$, so $\frac{dC}{dt} = 2\pi \frac{dr}{dt} = 2\pi \times \frac{2}{\pi} = 4$ cm per second.