

YEAR 2 PURE · CHAPTER 6

Geometric Series

Specification: Edexcel Pure 4.5, 4.6 — Sequences and series · AQA section D5, D6

KEY FACTS — THIS CHAPTER

- A **geometric** sequence multiplies by the same **common ratio** r each time: $u_n = ar^{n-1}$.
- The sum of the first n terms is $S_n = \frac{a(1-r^n)}{1-r}$, for $r \neq 1$.
- If $|r| < 1$ the series **converges**, and its **sum to infinity** is $S_\infty = \frac{a}{1-r}$.
- If $|r| \geq 1$ the partial sums never settle, so there is no sum to infinity.
- To find when a term or a sum passes a value, take logs. Dividing by $\log r$ when $0 < r < 1$ **reverses** the inequality.
- Growth or decay by a fixed percentage each time step is geometric: multiply by 1.05 for 5% growth, by 0.85 for a 15% fall.

Where an arithmetic sequence adds the same amount each time, a geometric sequence multiplies by the same amount. That is how compound interest, depreciation and population growth behave. It also leads to something surprising: when the ratio is small enough, adding up infinitely many terms gives a finite answer.

1. Geometric sequences

KEY RESULT

With first term a and common ratio r , the n th term is

$$u_n = ar^{n-1}$$

The ratio of any term to the one before is always r : $\frac{u_{n+1}}{u_n} = r$.

WORKED EXAMPLE 6.1 · TWO TERMS FIX THE SEQUENCE

The 3rd term of a geometric sequence is 18 and the 6th term is 486. Find the first term and the common ratio.

1. Write both terms using the formula:

$$ar^2 = 18$$

$$ar^5 = 486$$

2. Divide the second by the first: $r^3 = 27$, so $r = 3$.

3. Then $a = \frac{18}{9} = 2$.

WORKED EXAMPLE 6.2 · WHEN DOES A TERM PASS A VALUE?

The sequence 3, 6, 12, 24, ... is geometric. Find the first term that is greater than 10 000.

1. $a = 3$ and $r = 2$, so we need $3 \times 2^{n-1} > 10\,000$.
2. Take logs of both sides:

$$(n - 1) \log 2 > \log \frac{10\,000}{3}$$

$$n - 1 > 11.7 \text{ (3 s.f.)}$$

3. So $n = 13$. Check: $u_{12} = 6144$ and $u_{13} = 12\,288$.

2. Geometric series

THE SUM OF A GEOMETRIC SERIES

For $r \neq 1$, the sum of the first n terms is

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}$$

The two forms are equal. The second avoids negative signs when $r > 1$.

PROOF

Write out S_n , then multiply it by r :

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rS_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting, everything cancels except the first term of one line and the last term of the other: $S_n - rS_n = a - ar^n$. So $S_n(1 - r) = a(1 - r^n)$, and dividing by $1 - r$ gives the formula.

WORKED EXAMPLE 6.3 · THE SUM OF TEN TERMS

Find the sum of the first 10 terms of $2 + 6 + 18 + \dots$

Here $a = 2$ and $r = 3$, so

$$\begin{aligned} S_{10} &= \frac{2(3^{10} - 1)}{3 - 1} \\ &= 3^{10} - 1 = 59\,048 \end{aligned}$$

WORKED EXAMPLE 6.4 · A SUM IN SIGMA NOTATION

Evaluate $\sum_{k=1}^6 3 \times 2^k$.

The terms are 6, 12, 24, $\dots$: a geometric series with $a = 6$, $r = 2$ and 6 terms. So

$$\sum_{k=1}^6 3 \times 2^k = \frac{6(2^6 - 1)}{2 - 1} = 378$$

COMMON MISTAKE

Read the first term off the series, not the formula. In $\sum_{k=1}^6 3 \times 2^k$ the first term is $3 \times 2^1 = 6$, not 3, because the counter starts at $k = 1$.

3. The sum to infinity

When $|r| < 1$ the terms shrink towards 0, and the sums $S_1, S_2, S_3, \dots$ close in on a fixed value.

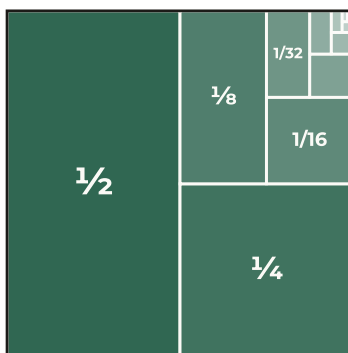
THE SUM TO INFINITY

A geometric series **converges** if and only if $|r| < 1$, and then

$$S_{\infty} = \frac{a}{1 - r}$$

WHY IT WORKS

In $S_n = \frac{a(1 - r^n)}{1 - r}$, the only part that changes with n is r^n . When $|r| < 1$, r^n gets as close to 0 as you like as n grows, so S_n gets as close as you like to $\frac{a}{1 - r}$. When $|r| \geq 1$, r^n doesn't shrink, and the sums never settle.



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ fills the square exactly: its sum to infinity is 1.

WORKED EXAMPLE 6.5 · A SUM TO INFINITY

Find the sum to infinity of $24 + 12 + 6 + \dots$

$a = 24$ and $r = \frac{1}{2}$, and $|r| < 1$, so

$$S_{\infty} = \frac{24}{1 - \frac{1}{2}} = 48$$

WORKED EXAMPLE 6.6 · A RECURRING DECIMAL AS A FRACTION

Write $0.272727\dots$ as a fraction in its simplest form.

1. The decimal is a geometric series:

$$0.2727\dots = \frac{27}{100} + \frac{27}{10\,000} + \dots$$

with $a = \frac{27}{100}$ and $r = \frac{1}{100}$.

2. So

$$\begin{aligned} S_{\infty} &= \frac{27/100}{1 - 1/100} \\ &= \frac{27}{99} = \frac{3}{11} \end{aligned}$$

WORKED EXAMPLE 6.7 · FINDING THE RATIO

A geometric series has first term 5 and sum to infinity 20. Find the common ratio.

$\frac{5}{1 - r} = 20$, so $1 - r = \frac{1}{4}$ and $r = \frac{3}{4}$. This has $|r| < 1$, as it must for the series to converge.

4. Modelling with geometric sequences

A quantity that rises or falls by the same **percentage** each year is multiplied by the same number each year, so its values form a geometric sequence.

WORKED EXAMPLE 6.8 · DEPRECIATION

A car is worth £18 000 when new, and loses 15% of its value each year. After how many whole years is it first worth less than £5000?

1. Each year the value is multiplied by 0.85, so after n years it is $18\,000 \times 0.85^n$. We need $18\,000 \times 0.85^n < 5000$.
2. Take logs. $\log 0.85$ is negative, so dividing by it reverses the inequality:

$$\begin{aligned} n \log 0.85 &< \log \frac{5000}{18\,000} \\ n &> \frac{\log(5/18)}{\log 0.85} \\ &= 7.88 \text{ (3 s.f.)} \end{aligned}$$

3. So after 8 years. Check: $18\,000 \times 0.85^7 = £5770$ and $18\,000 \times 0.85^8 = £4905$ (to the nearest pound).

COMMON MISTAKE

Dividing an inequality by a negative number flips it. For $0 < r < 1$, $\log r$ is negative, so from $n \log r < \log c$ you get $n > \frac{\log c}{\log r}$. Forgetting the flip gives “ $n < 7.88$ ”, which is the opposite of the truth.

THE MODEL'S LIMITS

A fixed percentage each year is a simplification. Real cars lose value fastest in their first year or two, and the model predicts a value that never quite reaches zero, which a scrapped car does.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

A geometric sequence begins 5, 15, 45, ...

- (a) State the common ratio. [1]
- (b) Find the 8th term. [1]
- (c) Find the sum of the first 8 terms. [1]

SOLUTION

(a) $r = \frac{15}{5} = 3.$

(b) $u_8 = 5 \times 3^7 = 10\,935.$

(c) $S_8 = \frac{5(3^8 - 1)}{3 - 1} = \frac{5 \times 6560}{2} = 16\,400.$

Question 2

[3 marks]

For the series $12 - 6 + 3 - \dots$

- (a) Explain why the series has a sum to infinity. [1]
- (b) Find the sum to infinity. [2]

SOLUTION

(a) The common ratio is $r = -\frac{1}{2}$, and $|r| < 1$, so the series converges.

(b)

$$S_{\infty} = \frac{12}{1 - (-\frac{1}{2})} = \frac{12}{\frac{3}{2}} = 8$$

Question 3

[2 marks]

Evaluate $\sum_{k=1}^5 2^k.$

SOLUTION

The terms are 2, 4, 8, 16, 32: a geometric series with $a = 2$, $r = 2$ and 5 terms.

$$\sum_{k=1}^5 2^k = \frac{2(2^5 - 1)}{2 - 1} = 62$$

Question 4

[3 marks]

Use a geometric series to write $0.454545 \dots$ as a fraction in its simplest form.

SOLUTION

$$0.4545 \dots = \frac{45}{100} + \frac{45}{10\,000} + \dots, \text{ with } a = \frac{45}{100} \text{ and } r = \frac{1}{100}:$$

$$S_{\infty} = \frac{45/100}{99/100} = \frac{45}{99} = \frac{5}{11}$$

Standard**Question 5**

[4 marks]

The 2nd term of a geometric sequence is 6 and the 5th term is 162. Find the first term, the common ratio, and the sum of the first 6 terms.

SOLUTION

$$ar = 6 \text{ and } ar^4 = 162. \text{ Dividing, } r^3 = 27, \text{ so } r = 3 \text{ and } a = 2.$$

$$S_6 = \frac{2(3^6 - 1)}{3 - 1} = 728$$

Question 6

[4 marks]

A geometric series has second term 4 and sum to infinity 16. Find the first term and the common ratio.

SOLUTION

$ar = 4$ and $\frac{a}{1-r} = 16$, so $a = 16(1-r)$. Substituting,

$$\begin{aligned} 16r(1-r) &= 4 \\ 4r^2 - 4r + 1 &= 0 \\ (2r-1)^2 &= 0 \end{aligned}$$

So $r = \frac{1}{2}$ and $a = 8$.

Question 7

[4 marks]

Find the smallest number of terms of the series $4 + 6 + 9 + \dots$ whose sum is greater than 1000.

SOLUTION

$a = 4$ and $r = \frac{3}{2}$, so $S_n = \frac{4(1.5^n - 1)}{0.5} = 8(1.5^n - 1)$. We need $8(1.5^n - 1) > 1000$, so $1.5^n > 126$:

$$n > \frac{\log 126}{\log 1.5} = 11.9 \text{ (3 s.f.)}$$

So $n = 12$. Check: $S_{11} = 684$ and $S_{12} = 1030$, to 3 s.f.

Question 8

[4 marks]

Prove that the sum of the first n terms of a geometric series with first term a and common ratio r , where $r \neq 1$, is $\frac{a(1-r^n)}{1-r}$.

SOLUTION

$$\begin{aligned} S_n &= a + ar + \dots + ar^{n-1} \\ rS_n &= ar + ar^2 + \dots + ar^n \end{aligned}$$

Subtracting the second line from the first, all but two terms cancel: $S_n - rS_n = a - ar^n$. So $S_n(1-r) = a(1-r^n)$, and since $r \neq 1$ we can divide by $1-r$:

$$S_n = \frac{a(1-r^n)}{1-r}$$

Question 9

[4 marks]

The series $1 + 2x + 4x^2 + 8x^3 + \dots$ is geometric.

- (a) Find the values of x for which the series converges. [2]
- (b) Find the sum to infinity when $x = \frac{1}{3}$. [2]

SOLUTION

- (a) The common ratio is $2x$. It converges when $|2x| < 1$, that is $-\frac{1}{2} < x < \frac{1}{2}$.
- (b) $r = \frac{2}{3}$, so $S_{\infty} = \frac{1}{1 - \frac{2}{3}} = 3$.

Exam style**Question 10**

[6 marks]

A car costs £22 000 new. Its value falls by 12% each year.

- (a) Find its value after 4 years, to the nearest pound. [2]
- (b) Find the number of whole years after which it is first worth less than £10 000. [3]
- (c) Give one reason why the model may not be accurate. [1]

SOLUTION

- (a) $22\,000 \times 0.88^4 = £13\,193$.
- (b) We need $22\,000 \times 0.88^n < 10\,000$, so $0.88^n < \frac{5}{11}$. $\log 0.88$ is negative, so dividing by it reverses the inequality:

$$n > \frac{\log(5/11)}{\log 0.88} = 6.17 \text{ (3 s.f.)}$$

After 7 years. Check: the value is £10 217 after 6 years and £8991 after 7.

- (c) For example, cars usually lose value faster in the first year or two, so a single fixed percentage won't match every year.

Question 11

[6 marks]

A runner runs 5 km in the first week of training, and increases the distance by 10% each week after that.

- (a) Find the distance she runs in the 8th week. [2]
- (b) Find the total distance run in the first 12 weeks. [2]
- (c) In which week does she first run more than 20 km? [2]

SOLUTION

(a) $5 \times 1.1^7 = 9.74$ km (3 s.f.).

(b)

$$S_{12} = \frac{5(1.1^{12} - 1)}{1.1 - 1} = 107 \text{ km (3 s.f.)}$$

(c) We need $5 \times 1.1^{n-1} > 20$, so $1.1^{n-1} > 4$ and

$$n - 1 > \frac{\log 4}{\log 1.1} = 14.5 \text{ (3 s.f.)}$$

So $n = 16$: she first runs more than 20 km in the 16th week.

Question 12

[8 marks]

The first three terms of a geometric series are k , $k - 6$ and $k - 9$.

- (a) Show that $k = 12$. [3]
- (b) Find the sum to infinity. [2]
- (c) Find the smallest value of n for which $S_{\infty} - S_n < 0.01$. [3]

SOLUTION

(a) The ratios of consecutive terms are equal, so $\frac{k-6}{k} = \frac{k-9}{k-6}$. Then

$$\begin{aligned}(k-6)^2 &= k(k-9) \\ k^2 - 12k + 36 &= k^2 - 9k \\ k &= 12\end{aligned}$$

(b) The series is $12 + 6 + 3 + \dots$, with $r = \frac{1}{2}$, so $S_{\infty} = \frac{12}{1 - \frac{1}{2}} = 24$.

(c) $S_n = 24 \left(1 - \left(\frac{1}{2}\right)^n\right)$, so $S_\infty - S_n = \frac{24}{2^n}$. We need $2^n > 2400$:

$$n > \frac{\log 2400}{\log 2} = 11.2 \text{ (3 s.f.)}$$

So $n = 12$.