

YEAR 2 PURE · CHAPTER 3

Functions

Specification: Edexcel Pure 2.8, 2.11 — Algebra and functions · AQA section B8, B11

KEY FACTS — THIS CHAPTER

- A **function** sends each input in its **domain** to exactly one output. The set of outputs is its **range**.
- A function is **one-to-one** if each output comes from only one input, and **many-to-one** otherwise.
- **Composite** functions: $fg(x) = f(g(x))$. Apply g first, then f . In general $fg \neq gf$.
- The **inverse** f^{-1} undoes f . Find it by writing $y = f(x)$ and rearranging for x . Only one-to-one functions have inverses.
- The domain of f^{-1} is the range of f , and the range of f^{-1} is the domain of f .
- The graph of $y = f^{-1}(x)$ is the **reflection** of $y = f(x)$ in the line $y = x$.

You have used $f(x)$ notation since Year 1. This chapter makes it precise. A function comes with a domain, the inputs it accepts, and a range, the outputs it can give. That precision is what lets you chain functions together and run them backwards.

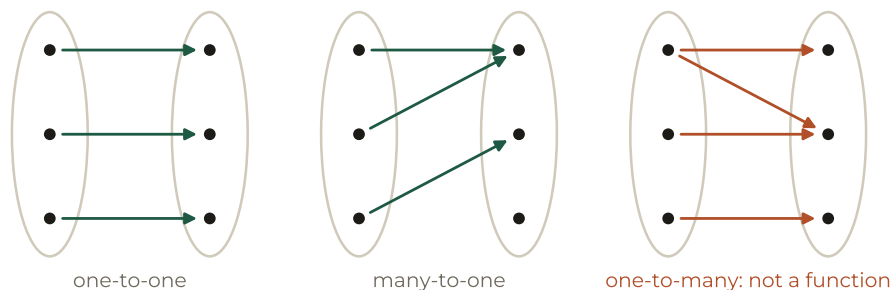
1. Domain and range

KEY RESULT

A **function** is a rule that gives exactly one output for each input.

- The **domain** is the set of inputs, written after the rule: $f(x) = x^2$, $x \in \mathbb{R}$.
- The **range** is the set of outputs. Describe it with $f(x)$, not x , as in $f(x) \geq 0$.
- A function is **one-to-one** if different inputs always give different outputs, and **many-to-one** if some output comes from more than one input.

A rule that gives some input **two** outputs, like $y = \pm\sqrt{x}$, is not a function at all.



One-to-one and many-to-one are both functions. One-to-many is not.

WORKED EXAMPLE 3.1 • THE RANGE OF A QUADRATIC

Find the range of $f(x) = x^2 - 4x + 7$, $x \in \mathbb{R}$.

1. Complete the square:

$$f(x) = (x - 2)^2 + 3$$

2. $(x - 2)^2 \geq 0$, and it equals 0 when $x = 2$. So the smallest output is 3, and every value above 3 is reached too.
3. The range is $f(x) \geq 3$.

WORKED EXAMPLE 3.2 • A RESTRICTED DOMAIN

Find the range of $g(x) = 2x + 1$, $-1 \leq x \leq 3$.

g is increasing, so its smallest and largest outputs come from the ends of the domain: $g(-1) = -1$ and $g(3) = 7$. The range is $-1 \leq g(x) \leq 7$.

COMMON MISTAKE

Write the range in terms of the **output**: $f(x) \geq 3$ or $y \geq 3$, never $x \geq 3$. The letter x belongs to the domain.

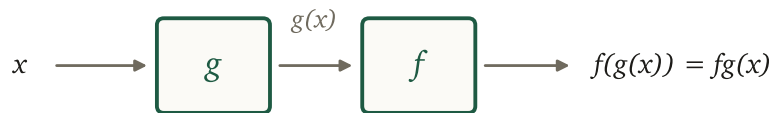
2. Composite functions

Feeding the output of one function into another makes a **composite** function.

KEY RESULT

$$fg(x) = f(g(x))$$

Work from the inside out: apply g **first**, then f . The order matters, and $fg(x)$ and $gf(x)$ are usually different.



$fg(x)$: x goes through g first, and g 's output goes through f .

WORKED EXAMPLE 3.3 • TWO ORDERS, TWO ANSWERS

$f(x) = 3x - 1$ and $g(x) = x^2 + 2$, both for $x \in \mathbb{R}$. Find $fg(x)$ and $gf(x)$, and solve $fg(x) = 32$.

1. For $fg(x)$, put $g(x)$ into f :

$$fg(x) = 3(x^2 + 2) - 1 = 3x^2 + 5$$

2. For $gf(x)$, put $f(x)$ into g :

$$\begin{aligned} gf(x) &= (3x - 1)^2 + 2 \\ &= 9x^2 - 6x + 3 \end{aligned}$$

3. Solve $3x^2 + 5 = 32$: $x^2 = 9$, so $x = 3$ or $x = -3$.

COMMON MISTAKE

$fg(x)$ does **not** mean f first. It reads right to left, like $f(g(x))$: the function nearest the x acts first.

3. Inverse functions

The inverse of a function runs it backwards: if f takes 2 to 7, then f^{-1} takes 7 back to 2.

KEY RESULT

To find $f^{-1}(x)$:

1. Write $y = f(x)$.
2. Rearrange to make x the subject.
3. Swap the letters to write the result as a function of x .

The domain of f^{-1} is the **range** of f , and the range of f^{-1} is the **domain** of f . Also $ff^{-1}(x) = f^{-1}f(x) = x$.

WORKED EXAMPLE 3.4 • THE INVERSE OF A FRACTION

$f(x) = \frac{2x+3}{x-1}$, $x > 1$. Find $f^{-1}(x)$ and state its domain.

1. Write $y = \frac{2x+3}{x-1}$ and clear the fraction:

$$\begin{aligned} y(x-1) &= 2x+3 \\ \Rightarrow xy - y &= 2x+3 \end{aligned}$$

2. Collect the x terms and factorise:

$$\begin{aligned} x(y-2) &= y+3 \\ \Rightarrow x &= \frac{y+3}{y-2} \end{aligned}$$

3. So $f^{-1}(x) = \frac{x+3}{x-2}$.

4. Its domain is the range of f . Since $f(x) = 2 + \frac{5}{x-1}$ and the fraction is positive for $x > 1$, the range of f is $f(x) > 2$. So the domain of f^{-1} is $x > 2$.

COMMON MISTAKE

$f^{-1}(x)$ is the inverse **function**, not the reciprocal. For $f(x) = 2x$, $f^{-1}(x) = \frac{1}{2}x$, but $\frac{1}{f(x)} = \frac{1}{2x}$.

A many-to-one function has no inverse: running it backwards would give one input two outputs. The fix is to **restrict the domain** until the function is one-to-one.

WORKED EXAMPLE 3.5 • RESTRICTING THE DOMAIN

$f(x) = x^2 - 4x + 7$, $x \geq 2$. Find $f^{-1}(x)$ and state its domain and range.

1. On $x \geq 2$ the function is one-to-one: it is the right half of the parabola, from its vertex $(2, 3)$. Its range is $f(x) \geq 3$.
2. Write $y = (x - 2)^2 + 3$ and rearrange:

$$\begin{aligned}(x - 2)^2 &= y - 3 \\ \Rightarrow x &= 2 + \sqrt{y - 3}\end{aligned}$$

Take the **positive** root, because $x \geq 2$.

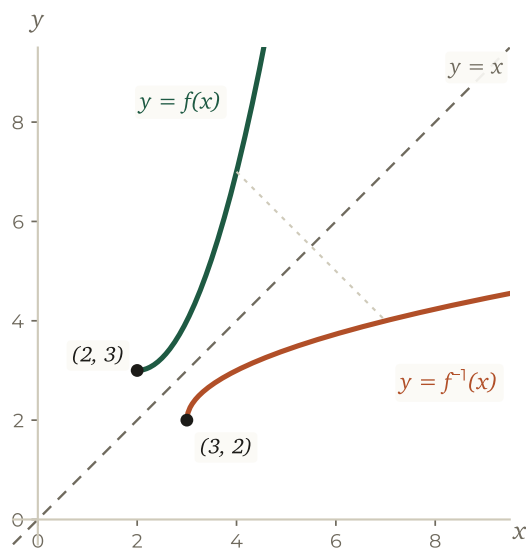
3. So $f^{-1}(x) = 2 + \sqrt{x - 3}$, with domain $x \geq 3$ and range $f^{-1}(x) \geq 2$.

THE GRAPH OF THE INVERSE

The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.

WHY IT WORKS

If the point (a, b) is on $y = f(x)$, then $f(a) = b$, so $f^{-1}(b) = a$ and the point (b, a) is on $y = f^{-1}(x)$. Swapping the coordinates of every point is exactly a reflection in $y = x$.



f and f^{-1} are mirror images in $y = x$: the point $(2, 3)$ becomes $(3, 2)$.

4. Functions in modelling

A function used as a model needs a domain that makes sense in the context, and its range then tells you what the model predicts.

WORKED EXAMPLE 3.6 · A THROWN BALL

A ball is thrown upwards from a height of 1.5 m. Its height after t seconds is modelled by $h(t) = 1.5 + 14t - 4.9t^2$. Find the greatest height, a suitable domain for h , and the range.

1. Complete the square:

$$h(t) = 11.5 - 4.9 \left(t - \frac{10}{7}\right)^2$$

so the greatest height is 11.5 m, when $t = \frac{10}{7} = 1.43$ s (3 s.f.).

2. The ball lands when $h(t) = 0$:

$$\begin{aligned} \left(t - \frac{10}{7}\right)^2 &= \frac{11.5}{4.9} \\ \Rightarrow t &= 2.96 \text{ s (3 s.f.)} \end{aligned}$$

taking the positive root, because the ball lands after it is thrown.

3. So a suitable domain is $0 \leq t \leq 2.96$, and the range is $0 \leq h(t) \leq 11.5$.

IN CONTEXT

The model says the ball rises to 11.5 m and lands about 3 seconds after it is thrown. Outside $0 \leq t \leq 2.96$ it predicts nonsense, such as negative heights, which is why the domain is part of the model. It also ignores air resistance, so the real ball won't go quite as high.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

$f(x) = 5 - 2x$ and $g(x) = x^2 + 1$, both for $x \in \mathbb{R}$. Find:

- (a) $fg(3)$
- (b) $gf(3)$
- (c) $ff(x)$, in its simplest form

SOLUTION

- (a) $g(3) = 10$, so $fg(3) = f(10) = 5 - 20 = -15$.
- (b) $f(3) = -1$, so $gf(3) = g(-1) = 2$.
- (c) $ff(x) = 5 - 2(5 - 2x) = 4x - 5$.

Question 2

[3 marks]

State whether each function is one-to-one or many-to-one.

- (a) $f(x) = 3x + 2$, $x \in \mathbb{R}$
- (b) $g(x) = x^2$, $x \in \mathbb{R}$
- (c) $h(x) = x^2$, $x \geq 0$

SOLUTION

- (a) One-to-one: a straight line with non-zero gradient never repeats an output.
- (b) Many-to-one: for example $g(2) = g(-2) = 4$.
- (c) One-to-one: with the negative inputs removed, each output comes from a single input.

Question 3

[3 marks]

Find the range of each function.

- (a) $f(x) = 4x - 1, 0 \leq x \leq 5$
- (b) $g(x) = x^2 + 6x + 13, x \in \mathbb{R}$

SOLUTION

- (a) f is increasing, with $f(0) = -1$ and $f(5) = 19$. The range is $-1 \leq f(x) \leq 19$.
- (b) $g(x) = (x + 3)^2 + 4$, and $(x + 3)^2 \geq 0$, so the range is $g(x) \geq 4$.

Question 4

[3 marks]

$f(x) = \frac{x+4}{3}, x \in \mathbb{R}$. Find $f^{-1}(x)$, and check that $ff^{-1}(x) = x$.

SOLUTION

Write $y = \frac{x+4}{3}$, so $3y = x + 4$ and $x = 3y - 4$. So

$$f^{-1}(x) = 3x - 4$$

Check: $ff^{-1}(x) = \frac{(3x-4)+4}{3} = \frac{3x}{3} = x. \checkmark$

Standard**Question 5**

[4 marks]

$f(x) = 2x + 3, x \in \mathbb{R}$, and $g(x) = \frac{1}{x-1}, x \neq 1$.

- (a) Find $gf(x)$ in its simplest form. [2]
- (b) Solve $gf(x) = \frac{1}{4}$. [2]

SOLUTION**(a)**

$$gf(x) = \frac{1}{(2x+3)-1} = \frac{1}{2x+2}$$

(b) $\frac{1}{2x+2} = \frac{1}{4}$ gives $2x+2 = 4$, so $x = 1$.

Question 6**[4 marks]**

$$f(x) = \sqrt{x-2}, \quad x \geq 2.$$

(a) State the range of f . **[1]**

(b) Find $f^{-1}(x)$ and state its domain. **[3]**

SOLUTION

(a) A square root is never negative, and $f(2) = 0$, so the range is $f(x) \geq 0$.

(b) Write $y = \sqrt{x-2}$. Squaring, $y^2 = x-2$, so $x = y^2 + 2$. So

$$f^{-1}(x) = x^2 + 2, \quad x \geq 0$$

The domain of f^{-1} is the range of f .

Question 7**[4 marks]**

$f(x) = \frac{3}{x+1}$, $x > -1$. Find $f^{-1}(x)$ and state its domain.

SOLUTION

Write $y = \frac{3}{x+1}$, so $x+1 = \frac{3}{y}$ and $x = \frac{3}{y} - 1$. So

$$f^{-1}(x) = \frac{3}{x} - 1$$

For $x > -1$ the denominator $x+1$ is positive, so $f(x) > 0$, and it takes every positive value. The domain of f^{-1} is $x > 0$.

Question 8

[5 marks]

$$f(x) = x^2 - 6x + 10, x \geq 3.$$

(a) Show that $f(x) = (x - 3)^2 + 1$, and state the range of f . [2]

(b) Find $f^{-1}(x)$, stating its domain. [3]

SOLUTION

(a) $(x - 3)^2 + 1 = x^2 - 6x + 9 + 1 = x^2 - 6x + 10$. ✓ Since $(x - 3)^2 \geq 0$, with 0 at $x = 3$, the range is $f(x) \geq 1$.

(b) Write $y = (x - 3)^2 + 1$. Then $(x - 3)^2 = y - 1$, and because $x \geq 3$ take the positive root: $x = 3 + \sqrt{y - 1}$. So

$$f^{-1}(x) = 3 + \sqrt{x - 1}, \quad x \geq 1$$

Question 9

[4 marks]

$f(x) = 2x - 5$ and $g(x) = x^2$, both for $x \in \mathbb{R}$. Solve $fg(x) = gf(x)$, giving your answers in surd form.

SOLUTION

$fg(x) = 2x^2 - 5$ and $gf(x) = (2x - 5)^2 = 4x^2 - 20x + 25$. Setting them equal:

$$\begin{aligned} 2x^2 - 20x + 30 &= 0 \\ \Rightarrow x^2 - 10x + 15 &= 0 \end{aligned}$$

By the quadratic formula,

$$x = \frac{10 \pm \sqrt{100 - 60}}{2} = 5 \pm \sqrt{10}$$

Exam style**Question 10**

[6 marks]

$$f(x) = \frac{2x + 1}{x - 3}, x > 3.$$

(a) Show that $f(x) = 2 + \frac{7}{x - 3}$, and hence state the range of f . [3]

(b) Find $f^{-1}(x)$, stating its domain. [3]

SOLUTION**(a)**

$$2 + \frac{7}{x-3} = \frac{2(x-3) + 7}{x-3} = \frac{2x+1}{x-3}$$

✓ For $x > 3$, $\frac{7}{x-3}$ takes every positive value, so the range is $f(x) > 2$.

(b) Write $y = \frac{2x+1}{x-3}$. Then $xy - 3y = 2x + 1$, so $x(y-2) = 3y+1$ and $x = \frac{3y+1}{y-2}$.

$$f^{-1}(x) = \frac{3x+1}{x-2}, \quad x > 2$$

Question 11**[6 marks]**

A company models its weekly profit, P thousand pounds, when it sells a product at a price of x pounds, by

$$P(x) = -2x^2 + 48x - 160$$

- (a) Write $P(x)$ in the form $a - b(x - c)^2$, where a , b and c are constants. **[2]**
- (b) State the greatest weekly profit the model predicts, and the price that gives it. **[2]**
- (c) The company only uses prices for which the model predicts a profit. Find a suitable domain for P . **[1]**
- (d) Give one limitation of the model. **[1]**

SOLUTION**(a)**

$$\begin{aligned} P(x) &= -2(x^2 - 24x) - 160 \\ &= -2((x - 12)^2 - 144) - 160 \\ &= 128 - 2(x - 12)^2 \end{aligned}$$

(b) The greatest profit is £128 000 a week, at a price of £12.

(c) $P(x) = 0$ when $(x - 12)^2 = 64$, so at $x = 4$ and $x = 20$. A suitable domain is $4 < x < 20$.

(d) For example, the model assumes the profit changes smoothly with price in exactly this quadratic way; in practice sales also depend on competitors' prices and demand, which the model ignores.

Question 12

[6 marks]

$f(x) = e^{2x} + 1$, $x \in \mathbb{R}$, and $g(x) = \ln x$, $x > 0$.

- (a) State the range of f . [1]
- (b) Find $f^{-1}(x)$, stating its domain. [3]
- (c) Solve $fg(x) = 10$. [2]

SOLUTION

(a) e^{2x} takes every positive value, so the range is $f(x) > 1$.

(b) Write $y = e^{2x} + 1$. Then $e^{2x} = y - 1$, so $2x = \ln(y - 1)$ and

$$f^{-1}(x) = \frac{1}{2} \ln(x - 1), \quad x > 1$$

(c) $fg(x) = e^{2\ln x} + 1 = e^{\ln x^2} + 1 = x^2 + 1$. So $x^2 + 1 = 10$ gives $x^2 = 9$. The domain of g is $x > 0$, so $x = 3$.