

YEAR 2 PURE • CHAPTER 12

Differentiation Rules

Specification: Edexcel Pure 7.1, 7.2, 7.4 — Differentiation • AQA section G1, G2, G4

KEY FACTS — THIS CHAPTER

- With x in radians, $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$.
- $\frac{d}{dx}(e^{kx}) = ke^{kx}$, $\frac{d}{dx}(a^x) = a^x \ln a$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$.
- **Chain rule:** if y is a function of u and u is a function of x , then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
- **Product rule:** $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$.
- **Quotient rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$.
- $\frac{d}{dx}(\tan x) = \sec^2 x$, and $\frac{dx}{dy} = 1 \div \frac{dy}{dx}$.

In Year 1 you could differentiate powers of x . This chapter adds the other functions, sines, exponentials and logarithms, and three rules for combining them: one for a function of a function, one for a product and one for a quotient. Together they differentiate almost anything you can write down.

1. sin and cos from first principles

The proof uses two limits. Both come from the small-angle approximations of Chapter 8, which is why the angle must be in radians.

KEY RESULT

With h in radians, as $h \rightarrow 0$,

$$\frac{\sin h}{h} \rightarrow 1 \quad \text{and} \quad \frac{\cos h - 1}{h} \rightarrow 0$$

For small h , $\sin h \approx h$, and $\cos h - 1 \approx -\frac{1}{2}h^2$, so the second fraction is about $-\frac{1}{2}h$.

WORKED EXAMPLE 12.1 · THE DERIVATIVE OF SIN FROM FIRST PRINCIPLES

Prove from first principles that the derivative of $\sin x$ is $\cos x$.

1. Expand $\sin(x + h)$ with the addition formula and collect terms:

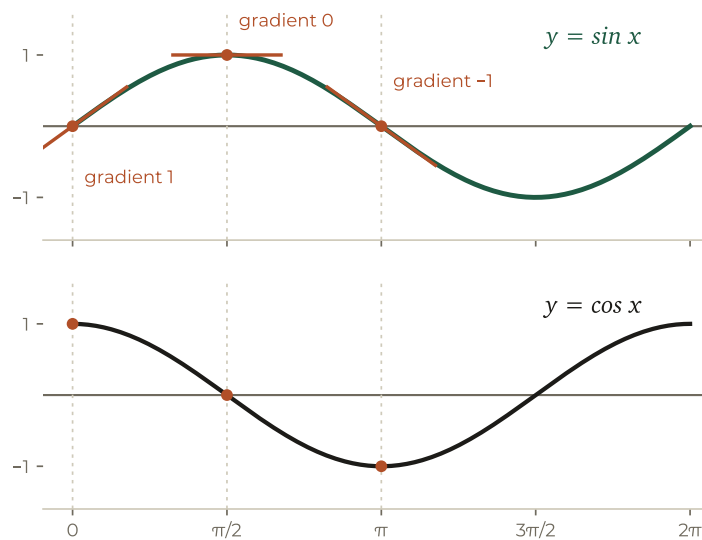
$$\begin{aligned}\sin(x + h) - \sin x &= \sin x \cos h + \cos x \sin h - \sin x \\ &= \sin x(\cos h - 1) + \cos x \sin h\end{aligned}$$

2. Divide by h :

$$\begin{aligned}\frac{\sin(x + h) - \sin x}{h} &= \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}\end{aligned}$$

3. As $h \rightarrow 0$, the limits above turn this into $\sin x \times 0 + \cos x \times 1 = \cos x$.

The same method with $\cos(x + h) = \cos x \cos h - \sin x \sin h$ shows that the derivative of $\cos x$ is $-\sin x$.



The gradient of $\sin x$ (top) at each point is the height of $\cos x$ (bottom) there.

COMMON MISTAKE

These derivatives are only true in radians. In degrees the gradient of $\sin x$ at $x = 0$ is $\frac{\pi}{180}$, not 1, and every result in this chapter would need an extra factor.

2. Exponentials and logarithms**KEY RESULT**

$$\begin{aligned}\frac{d}{dx}(e^{kx}) &= ke^{kx} \\ \frac{d}{dx}(a^x) &= a^x \ln a \\ \frac{d}{dx}(\ln x) &= \frac{1}{x}\end{aligned}$$

The second follows from the first, because $a = e^{\ln a}$, so $a^x = e^{x \ln a}$: that is e^{kx} with $k = \ln a$. Section 6 shows where the third comes from. Since $\ln(kx) = \ln k + \ln x$, the derivative of $\ln(kx)$ is also $\frac{1}{x}$.

WORKED EXAMPLE 12.2 · STANDARD DERIVATIVES

Differentiate $y = 3e^{2x} - 5 \ln x + 2^x$, and find the gradient of the curve at $x = 1$.

1. Differentiate term by term:

$$\frac{dy}{dx} = 6e^{2x} - \frac{5}{x} + 2^x \ln 2$$

2. At $x = 1$:

$$\begin{aligned}\frac{dy}{dx} &= 6e^2 - 5 + 2 \ln 2 \\ &= 40.7 \text{ (3 s.f.)}\end{aligned}$$

3. The chain rule

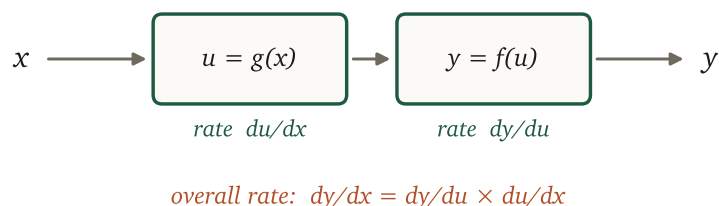
$y = (3x^2 - 1)^5$ is a function of a function: first work out $u = 3x^2 - 1$, then $y = u^5$. A small change in x changes u at the rate $\frac{du}{dx}$, and each change in u changes y at the rate $\frac{dy}{du}$, so the rates multiply.

KEY RESULT

If y is a function of u , and u is a function of x , then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

In words: differentiate the outside, keeping the inside as it is, then multiply by the derivative of the inside.



Two machines in a chain: the overall rate is the product of the two rates.

WORKED EXAMPLE 12.3 · A POWER OF A BRACKET

Differentiate $y = (3x^2 - 1)^5$.

1. Let $u = 3x^2 - 1$, so $y = u^5$. Then $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = 6x$.
2. Multiply, and write the answer in terms of x :

$$\begin{aligned}\frac{dy}{dx} &= 5(3x^2 - 1)^4 \times 6x \\ &= 30x(3x^2 - 1)^4\end{aligned}$$

WORKED EXAMPLE 12.4 · CHAINS WITH THE NEW FUNCTIONS

Differentiate (a) e^{x^2} , (b) $\ln(\cos x)$ and (c) $\sin^3 x$.

Each time, differentiate the outside function, then multiply by the derivative of the inside.

(a) The outside is e^u and the inside is x^2 : $e^{x^2} \times 2x = 2xe^{x^2}$.

(b) The outside is $\ln u$ and the inside is $\cos x$: $\frac{1}{\cos x} \times (-\sin x) = -\tan x$.

(c) $\sin^3 x$ means $(\sin x)^3$, so the outside is the cube: $3\sin^2 x \times \cos x = 3\sin^2 x \cos x$.

A special case comes up so often that it's worth knowing on its own: if the inside is kx , the chain rule just multiplies by k . So $\frac{d}{dx}(\sin kx) = k \cos kx$ and $\frac{d}{dx}(\cos kx) = -k \sin kx$.

4. The product rule

KEY RESULT

If u and v are functions of x ,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Think of uv as the area of a rectangle with sides u and v . When x grows a little, u grows by δu and v by δv , and the area gains three pieces: $u \delta v$ along one side, $v \delta u$ along the other, and a tiny corner $\delta u \delta v$. Divide by δx and let $\delta x \rightarrow 0$: the corner vanishes, and what's left is the product rule.

WORKED EXAMPLE 12.5 · STATIONARY POINTS OF A PRODUCT

Find the x -coordinates of the stationary points of $y = x^2 e^{3x}$.

1. With $u = x^2$ and $v = e^{3x}$, $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3e^{3x}$:

$$\begin{aligned} \frac{dy}{dx} &= x^2 \times 3e^{3x} + e^{3x} \times 2x \\ &= xe^{3x}(3x + 2) \end{aligned}$$

2. e^{3x} is never 0, so $\frac{dy}{dx} = 0$ when $x = 0$ or $x = -\frac{2}{3}$.

5. The quotient rule

KEY RESULT

If u and v are functions of x ,

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

The order on top matters, because it's a subtraction: start with v .

WORKED EXAMPLE 12.6 · THE DERIVATIVE OF TAN

Show that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Write $\tan x = \frac{\sin x}{\cos x}$, with $u = \sin x$ and $v = \cos x$:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

The reciprocal functions follow in the same way, from the quotient rule or the chain rule:

THE OTHER TRIG DERIVATIVES

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \sec^2 x \\ \frac{d}{dx}(\sec x) &= \sec x \tan x \\ \frac{d}{dx}(\operatorname{cosec} x) &= -\operatorname{cosec} x \cot x \\ \frac{d}{dx}(\cot x) &= -\operatorname{cosec}^2 x \end{aligned}$$

WORKED EXAMPLE 12.7 · A MAXIMUM

Find the coordinates of the stationary point of $y = \frac{\ln x}{x}$, $x > 0$, and show that it is a maximum.

1. With $u = \ln x$ and $v = x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2} \\ &= \frac{1 - \ln x}{x^2}\end{aligned}$$

2. This is 0 when $\ln x = 1$, so $x = e$ and $y = \frac{1}{e}$.
3. x^2 is positive, so $\frac{dy}{dx}$ has the sign of $1 - \ln x$: positive for $x < e$ and negative for $x > e$. The curve rises then falls, so $(e, \frac{1}{e})$ is a maximum.

6. Using dx/dy**KEY RESULT**

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

So when x is given as a function of y , find $\frac{dx}{dy}$ and turn it upside down.

WORKED EXAMPLE 12.8 · TURNING THE GRADIENT UPSIDE DOWN

A curve has equation $x = y^3 + y$. Find the gradient of the curve at the point $(2, 1)$.

1. $\frac{dx}{dy} = 3y^2 + 1$, which is 4 when $y = 1$.
2. So $\frac{dy}{dx} = \frac{1}{4}$ at $(2, 1)$.

WHERE THE DERIVATIVE OF LN COMES FROM

If $y = \ln x$, then $x = e^y$, so $\frac{dx}{dy} = e^y = x$. Turning it upside down gives $\frac{dy}{dx} = \frac{1}{x}$.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[4 marks]

Differentiate each of these with respect to x .

(a) e^{5x} [1]

(b) $4 \ln x$ [1]

(c) 3^x [1]

(d) $\sin 4x$ [1]

SOLUTION

(a) $5e^{5x}$ (b) $\frac{4}{x}$ (c) $3^x \ln 3$ (d) $4 \cos 4x$

Question 2

[2 marks]

Differentiate $y = (2x + 1)^6$.

SOLUTION

By the chain rule, with inside $2x + 1$:

$$\frac{dy}{dx} = 6(2x + 1)^5 \times 2 = 12(2x + 1)^5$$

Question 3

[2 marks]

Differentiate $y = x^3 \ln x$.

SOLUTION

By the product rule, with $u = x^3$ and $v = \ln x$:

$$\begin{aligned}\frac{dy}{dx} &= x^3 \times \frac{1}{x} + \ln x \times 3x^2 \\ &= x^2(1 + 3 \ln x)\end{aligned}$$

Question 4

[2 marks]

Differentiate $y = \frac{x}{x+1}$.

SOLUTION

By the quotient rule, with $u = x$ and $v = x + 1$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+1) \times 1 - x \times 1}{(x+1)^2} \\ &= \frac{1}{(x+1)^2}\end{aligned}$$

Standard**Question 5**

[3 marks]

Find the gradient of $y = e^{\sin x}$ at the point where $x = \pi$.

SOLUTION

By the chain rule, $\frac{dy}{dx} = \cos x e^{\sin x}$. At $x = \pi$ this is $\cos \pi \times e^0 = -1$.

Question 6

[6 marks]

Find the coordinates of the stationary points of $y = x^2 e^{-x}$, and determine their nature.

SOLUTION

By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= x^2 \times (-e^{-x}) + e^{-x} \times 2x \\ &= xe^{-x}(2 - x)\end{aligned}$$

which is 0 when $x = 0$ or $x = 2$: the points $(0, 0)$ and $(2, 4e^{-2})$.

Differentiating again, $\frac{d^2y}{dx^2} = e^{-x}(x^2 - 4x + 2)$. At $x = 0$ this is $2 > 0$, so $(0, 0)$ is a minimum. At $x = 2$ it is $-2e^{-2} < 0$, so $(2, 4e^{-2})$ is a maximum.

Question 7

[3 marks]

Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

SOLUTION

Write $\sec x = (\cos x)^{-1}$ and use the chain rule:

$$\begin{aligned}\frac{d}{dx}(\cos x)^{-1} &= -(\cos x)^{-2} \times (-\sin x) \\ &= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x\end{aligned}$$

Question 8

[3 marks]

A curve has equation $x = y^2 - 4y$. Find the gradient of the curve at the point $(-3, 1)$.

SOLUTION

$\frac{dx}{dy} = 2y - 4$, which is -2 when $y = 1$. So $\frac{dy}{dx} = -\frac{1}{2}$ at $(-3, 1)$.

Question 9

[4 marks]

Find the equation of the tangent to $y = \ln(2x - 3)$ at the point where $x = 2$.

SOLUTION

When $x = 2$, $y = \ln 1 = 0$. By the chain rule, $\frac{dy}{dx} = \frac{2}{2x - 3}$, which is 2 when $x = 2$. The tangent is $y - 0 = 2(x - 2)$, that is $y = 2x - 4$.

Exam style**Question 10**

[6 marks]

$$f(x) = \frac{x^2 + 3}{x - 1}, \quad x > 1$$

(a) Show that $f'(x) = \frac{(x-3)(x+1)}{(x-1)^2}$. [3]

(b) Find the coordinates of the stationary point of $y = f(x)$, and show that it is a minimum. [3]

SOLUTION

(a) By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x-1) \times 2x - (x^2+3) \times 1}{(x-1)^2} \\ &= \frac{x^2 - 2x - 3}{(x-1)^2} = \frac{(x-3)(x+1)}{(x-1)^2} \end{aligned}$$

(b) For $x > 1$, $f'(x) = 0$ only when $x = 3$, and $f(3) = \frac{12}{2} = 6$. The denominator and $x + 1$ are positive, so $f'(x)$ has the sign of $x - 3$: negative just before $x = 3$ and positive just after. So $(3, 6)$ is a minimum.

Question 11

[5 marks]

Given that $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cos h - 1}{h} \rightarrow 0$ as $h \rightarrow 0$, prove from first principles that the derivative of $\cos x$ is $-\sin x$.

SOLUTION

Using the addition formula,

$$\begin{aligned} &\cos(x+h) - \cos x \\ &= \cos x \cos h - \sin x \sin h - \cos x \\ &= \cos x(\cos h - 1) - \sin x \sin h \end{aligned}$$

Dividing by h ,

$$\begin{aligned} &\frac{\cos(x+h) - \cos x}{h} \\ &= \cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \end{aligned}$$

As $h \rightarrow 0$ this tends to $\cos x \times 0 - \sin x \times 1 = -\sin x$.

Question 12

[7 marks]

A hot drink cools so that its temperature t minutes after it is made is $T = 20 + 70e^{-0.05t}$ °C.

- (a) State its temperature when it is made. [1]
- (b) Find the rate at which it is cooling after 10 minutes. [3]
- (c) Find when it is cooling at 1 °C per minute, and its temperature then. [3]

SOLUTION

(a) At $t = 0$, $T = 20 + 70 = 90$ °C.

(b) $\frac{dT}{dt} = 70 \times (-0.05)e^{-0.05t} = -3.5e^{-0.05t}$. At $t = 10$ this is $-3.5e^{-0.5} = -2.12$ (3 s.f.), so it is cooling at 2.12 °C per minute.

(c) $3.5e^{-0.05t} = 1$ gives $e^{-0.05t} = \frac{1}{3.5}$, so

$$t = \frac{\ln 3.5}{0.05} = 25.1 \text{ minutes (3 s.f.)}$$

Then $T = 20 + 70 \times \frac{1}{3.5} = 40$ °C.