

YEAR 2 PURE · CHAPTER 16

Areas and Differential Equations

Specification: Edexcel Pure 7.6, 8.3, 8.4, 8.7, 8.8 — Differentiation and integration · AQA section G6, H3, H4, H7, H8

KEY FACTS — THIS CHAPTER

- The area between $y = f(x)$ above and $y = g(x)$ below, from $x = a$ to $x = b$, is $\int_a^b (f(x) - g(x)) dx$.
- Under a parametric curve, the area is $\int y \frac{dx}{dt} dt$, with limits that are values of t .
- An integral is the limit of a sum of thin strips: $\int_a^b y dx = \lim_{\delta x \rightarrow 0} \sum_{x=a}^b y \delta x$.
- A **differential equation** links a rate of change to the quantity itself. “Proportional to” means “equals a constant k times”.
- To solve a **separable** equation $\frac{dy}{dx} = f(x)g(y)$, write $\int \frac{1}{g(y)} dy = \int f(x) dx$.
- The **general solution** contains a constant. A **particular solution** uses a given value to fix it.

This chapter puts the integration methods to work. The first half finds areas: between two curves, under a parametric curve, and as the limit of a sum of thin strips. The second half solves **differential equations**, where the unknown is a function and the equation says how fast it changes. They model anything whose rate of change depends on its current state, from cooling tea to growing populations.

1. The area between two curves

KEY RESULT

If $f(x) \geq g(x)$ for $a \leq x \leq b$, the area between the curves is

$$\int_a^b (f(x) - g(x)) dx$$

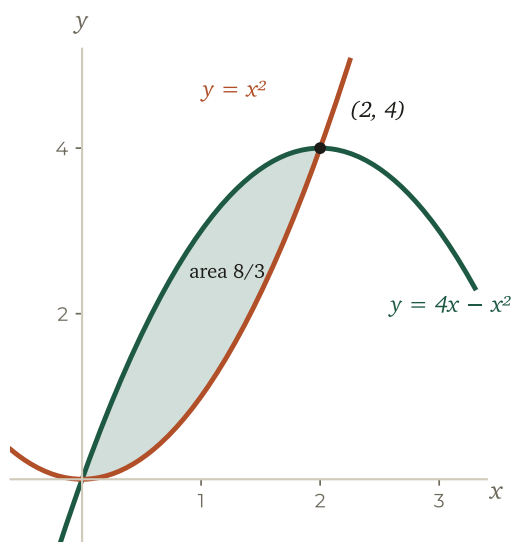
Subtracting first means that parts below the x -axis take care of themselves. To find a and b when they aren't given, solve $f(x) = g(x)$.

WORKED EXAMPLE 16.1 · TWO PARABOLAS

Find the area enclosed by the curves $y = 4x - x^2$ and $y = x^2$.

1. They meet where $x^2 = 4x - x^2$, so $2x^2 - 4x = 0$ and $x = 0$ or 2 . Between these, $4x - x^2$ is the upper curve.
2. So the area is

$$\begin{aligned} & \int_0^2 (4x - 2x^2) dx \\ &= \left[2x^2 - \frac{2}{3}x^3 \right]_0^2 = 8 - \frac{16}{3} = \frac{8}{3} \end{aligned}$$



The region between $y = 4x - x^2$ (above) and $y = x^2$ (below), from $x = 0$ to $x = 2$.

WORKED EXAMPLE 16.2 · BETWEEN SIN AND COS

Find the area enclosed between $y = \sin x$ and $y = \cos x$ from $x = \frac{\pi}{4}$ to $x = \frac{5\pi}{4}$.

The curves cross at both limits, and $\sin x \geq \cos x$ between them:

$$\begin{aligned} & \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx \\ &= \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4} \\ &= \sqrt{2} - (-\sqrt{2}) = 2\sqrt{2} \end{aligned}$$

Part of this region lies below the x -axis, but subtracting the curves first counts all of it correctly.

2. The area under a parametric curve

For a curve $x = x(t)$, $y = y(t)$, the area under it is still $\int y dx$. The chain rule turns dx into $\frac{dx}{dt} dt$, so you can integrate with respect to t without finding the Cartesian equation.

KEY RESULT

$$\text{Area} = \int_{t_1}^{t_2} y \frac{dx}{dt} dt$$

where t_1 and t_2 are the values of t at the ends of the region.

WORKED EXAMPLE 16.3 · A PARABOLA IN PARAMETRIC FORM

The curve $x = t^2$, $y = 2t$ runs from the origin to the point $(4, 4)$ as t goes from 0 to 2. Find the area under this part of the curve.

$\frac{dx}{dt} = 2t$, so

$$\int_0^2 2t \times 2t dt = \left[\frac{4}{3} t^3 \right]_0^2 = \frac{32}{3}$$

As a check, the Cartesian curve is $y = 2\sqrt{x}$, and $\int_0^4 2x^{\frac{1}{2}} dx = \frac{4}{3} \times 8 = \frac{32}{3}$ too.

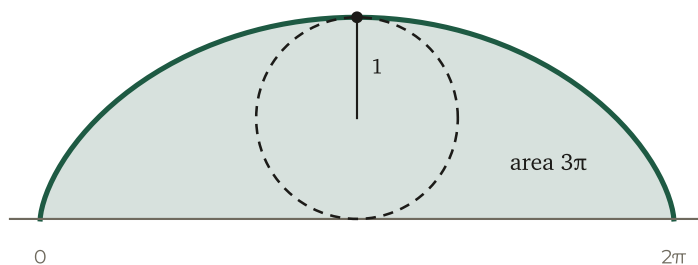
WORKED EXAMPLE 16.4 · THE AREA UNDER A CYCLOID

A point on the rim of a rolling wheel traces the cycloid $x = t - \sin t$, $y = 1 - \cos t$. Find the area under one arch, $0 \leq t \leq 2\pi$.

1. $\frac{dx}{dt} = 1 - \cos t$, so the area is $\int_0^{2\pi} (1 - \cos t)^2 dt$.
2. Expand, and use $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$:

$$\begin{aligned} & \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t \right) dt \\ &= \left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t \right]_0^{2\pi} \\ &= 3\pi \end{aligned}$$

The wheel has radius 1, so the arch has exactly three times the wheel's area.



One arch of the cycloid traced by a wheel of radius 1. The area under it is 3π , three times the wheel's area.

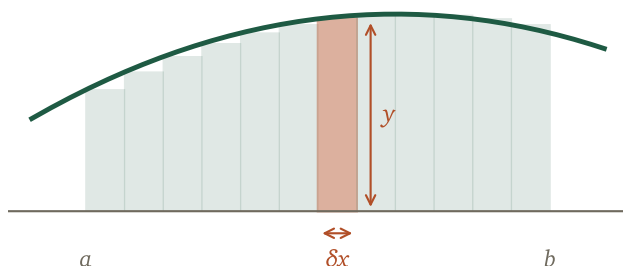
3. Integration as the limit of a sum

Split the region under a curve into thin strips of width δx . A strip at x is nearly a rectangle of height y , so its area is about $y \delta x$. Adding all the strips and letting their width shrink to zero gives the exact area.

KEY RESULT

$$\int_a^b y \, dx = \lim_{\delta x \rightarrow 0} \sum_{x=a}^b y \, \delta x$$

The integral sign is a stretched S, for “sum”, and dx is what δx becomes in the limit.



One strip: height y , width δx , area about $y \, \delta x$. The integral adds them up as $\delta x \rightarrow 0$.

WORKED EXAMPLE 16.5 · RECOGNISING A LIMIT OF A SUM

Find $\lim_{\delta x \rightarrow 0} \sum_{x=1}^3 (2x + 1) \, \delta x$.

This is the integral of $2x + 1$ from 1 to 3:

$$\begin{aligned} \int_1^3 (2x + 1) \, dx &= \left[x^2 + x \right]_1^3 \\ &= 12 - 2 = 10 \end{aligned}$$

4. Setting up differential equations

A **differential equation** is an equation involving a derivative. In a model it usually says how fast a quantity changes, given its current value.

KEY RESULT

- “The rate of change of P is proportional to P ” means $\frac{dP}{dt} = kP$.
- “... inversely proportional to r ” means $\frac{dr}{dt} = \frac{k}{r}$.
- A quantity that **decreases** has a negative rate, so write $-k$ with $k > 0$.

WORKED EXAMPLE 16.6 · NEWTON'S LAW OF COOLING

A cup of tea cools in a room at 20°C . Its temperature $T^\circ\text{C}$ falls at a rate proportional to the difference between T and the room temperature. Write down a differential equation for T .

The difference is $T - 20$, and T is falling, so

$$\frac{dT}{dt} = -k(T - 20)$$

where k is a positive constant and t is the time.

5. Solving separable differential equations

If $\frac{dy}{dx}$ is a function of x times a function of y , the variables can be **separated**: everything with y goes with dy , and everything with x goes with dx .

KEY RESULT

$$\begin{aligned} \frac{dy}{dx} &= f(x)g(y) \\ \Rightarrow \int \frac{1}{g(y)} dy &= \int f(x) dx \end{aligned}$$

One constant of integration is enough: put it on the x side. That gives the **general solution**, a family of curves. A given point on the curve picks out one of them, the **particular solution**.

WORKED EXAMPLE 16.7 · A PARTICULAR SOLUTION

Solve $\frac{dy}{dx} = xy$, given that $y = 2$ when $x = 0$.

1. Separate the variables and integrate:

$$\int \frac{1}{y} dy = \int x dx$$

$$\ln |y| = \frac{1}{2}x^2 + c$$

2. Take exponentials: $y = Ae^{\frac{1}{2}x^2}$, where $A = \pm e^c$.
3. $y = 2$ when $x = 0$ gives $A = 2$, so $y = 2e^{\frac{1}{2}x^2}$.

WORKED EXAMPLE 16.8 · SOLVING THE COOLING MODEL

The tea in Example 16.6 is at 90 °C when it is made, and at 55 °C ten minutes later. Find T in terms of t , and the time at which it reaches 30 °C.

1. Separate and integrate:

$$\int \frac{1}{T-20} dT = \int -k dt$$

$$\ln(T-20) = -kt + c$$

$$\text{so } T = 20 + Ae^{-kt}.$$

2. At $t = 0$, $90 = 20 + A$, so $A = 70$. At $t = 10$, $55 = 20 + 70e^{-10k}$, so $e^{-10k} = \frac{1}{2}$ and $k = \frac{\ln 2}{10} = 0.0693$ (3 s.f.). So $T = 20 + 70e^{-0.0693t}$.
3. $T = 30$ when $70e^{-kt} = 10$, so

$$t = \frac{\ln 7}{k} = \frac{10 \ln 7}{\ln 2}$$

$$= 28.1 \text{ minutes (3 s.f.)}$$

READING THE MODEL

As t grows, $e^{-kt} \rightarrow 0$, so $T \rightarrow 20$: the tea cools towards room temperature but never quite reaches it. The model assumes the room stays at 20 °C and that nothing else, such as stirring or adding milk, changes how fast the tea loses heat.

COMMON MISTAKE

Add the constant as soon as you integrate, not after rearranging. Writing $\ln y = \frac{1}{2}x^2$ and then adding c at the end gives $y = e^{\frac{1}{2}x^2} + c$, which is wrong: the constant should multiply, as A does in Example 16.7.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

Find the area enclosed between $y = x$ and $y = x^2$.

SOLUTION

They meet where $x^2 = x$, at $x = 0$ and $x = 1$, and $x \geq x^2$ between them:

$$\int_0^1 (x - x^2) dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = \frac{1}{6}$$

Question 2

[2 marks]

Find $\lim_{\delta x \rightarrow 0} \sum_{x=0}^3 x^2 \delta x$.

SOLUTION

$$\int_0^3 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^3 = 9$$

Question 3

[2 marks]

Write down a differential equation for each situation.

- (a) A population P grows at a rate proportional to its size. [1]
- (b) A liquid cools at a rate proportional to the difference between its temperature θ and 15°C . [1]

SOLUTION

- (a) $\frac{dP}{dt} = kP$, with $k > 0$.
- (b) $\frac{d\theta}{dt} = -k(\theta - 15)$, with $k > 0$.

Question 4

[3 marks]

Find the general solution of $\frac{dy}{dx} = 3x^2y$.

SOLUTION

$$\int \frac{1}{y} dy = \int 3x^2 dx$$
$$\ln |y| = x^3 + c$$

so $y = Ae^{x^3}$.

Standard**Question 5**

[3 marks]

A curve has parametric equations $x = 3t^2$, $y = 2t$. Find the area under the curve between $t = 0$ and $t = 1$.

SOLUTION

$\frac{dx}{dt} = 6t$, so the area is

$$\int_0^1 2t \times 6t dt = \left[4t^3 \right]_0^1 = 4$$

Question 6

[4 marks]

Find the area enclosed between $y = x^2 - 2x$ and $y = x$.

SOLUTION

They meet where $x^2 - 2x = x$, so $x(x - 3) = 0$: $x = 0$ or 3 . Between them the line is above the curve:

$$\begin{aligned}\int_0^3 (3x - x^2) dx &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^3 \\ &= \frac{27}{2} - 9 = \frac{9}{2}\end{aligned}$$

Question 7

[5 marks]

Solve $\frac{dy}{dx} = e^{x-y}$, given that $y = \ln 2$ when $x = 0$. Give y in terms of x .

SOLUTION

$e^{x-y} = e^x e^{-y}$, so the variables separate:

$$\begin{aligned}\int e^y dy &= \int e^x dx \\ e^y &= e^x + c\end{aligned}$$

At $x = 0$, $2 = 1 + c$, so $c = 1$. Then $e^y = e^x + 1$ and $y = \ln(e^x + 1)$.

Question 8

[4 marks]

Solve $\frac{dy}{dx} = \frac{\cos x}{y}$, given that $y = 2$ when $x = 0$.

SOLUTION

$$\begin{aligned}\int y dy &= \int \cos x dx \\ \frac{1}{2}y^2 &= \sin x + c\end{aligned}$$

At $x = 0$, $2 = 0 + c$, so $y^2 = 2 \sin x + 4$. Since $y = 2 > 0$ at the start, take the positive root:
 $y = \sqrt{2 \sin x + 4}$.

Question 9

[5 marks]

The ellipse $x = 2 \cos t$, $y = 3 \sin t$ is traced as t goes from 0 to 2π .

- (a) Show that the area of the quarter of the ellipse in the first quadrant is $\int_0^{\pi/2} 6 \sin^2 t \, dt$. [3]
- (b) Hence find the area of the whole ellipse. [2]

SOLUTION

- (a) In the first quadrant, x runs from 0 to 2 as t runs from $\frac{\pi}{2}$ down to 0. With $\frac{dx}{dt} = -2 \sin t$,

$$\begin{aligned} \int_{\pi/2}^0 3 \sin t \times (-2 \sin t) \, dt \\ = \int_{\pi/2}^0 -6 \sin^2 t \, dt = \int_0^{\pi/2} 6 \sin^2 t \, dt \end{aligned}$$

- (b) $\int_0^{\pi/2} 3(1 - \cos 2t) \, dt = \left[3t - \frac{3}{2} \sin 2t \right]_0^{\pi/2} = \frac{3\pi}{2}$, so the whole ellipse has area $4 \times \frac{3\pi}{2} = 6\pi$.

Exam style**Question 10**

[7 marks]

A liquid is made at 80°C and cools in a room at 15°C . Its temperature after t minutes is $\theta^\circ\text{C}$, where $\frac{d\theta}{dt} = -k(\theta - 15)$. After 5 minutes it is at 60°C .

- (a) Show that $\theta = 15 + 65e^{-kt}$. [3]
- (b) Find k , to 3 significant figures. [2]
- (c) Find the time at which the liquid reaches 25°C . [2]

SOLUTION

- (a) Separate the variables:

$$\begin{aligned} \int \frac{1}{\theta - 15} \, d\theta &= \int -k \, dt \\ \ln(\theta - 15) &= -kt + c \end{aligned}$$

So $\theta = 15 + Ae^{-kt}$, and $\theta = 80$ at $t = 0$ gives $A = 65$.

- (b) $60 = 15 + 65e^{-5k}$, so $e^{-5k} = \frac{9}{13}$ and $k = \frac{1}{5} \ln \frac{13}{9} = 0.0735$ (3 s.f.).

- (c) $65e^{-kt} = 10$, so

$$t = \frac{\ln 6.5}{k} = 25.5 \text{ minutes (3 s.f.)}$$

Question 11

[8 marks]

A rumour spreads through a school. The proportion x of students who have heard it after t days is modelled by $\frac{dx}{dt} = x(1 - x)$, and at $t = 0$ half the students have heard it.

- (a) Express $\frac{1}{x(1 - x)}$ in partial fractions. [2]
- (b) Hence show that $x = \frac{1}{1 + e^{-t}}$. [5]
- (c) Describe what happens to x in the long run. [1]

SOLUTION

(a) $\frac{1}{x(1 - x)} = \frac{1}{x} + \frac{1}{1 - x}$.

(b) Separate and integrate:

$$\int \left(\frac{1}{x} + \frac{1}{1 - x} \right) dx = \int dt$$

$$\ln x - \ln(1 - x) = t + c$$

(no modulus signs are needed, since $0 < x < 1$). So $\frac{x}{1 - x} = Ae^t$. At $t = 0$, $x = \frac{1}{2}$ gives $A = 1$. Then $x = e^t(1 - x)$, so $x = \frac{e^t}{1 + e^t} = \frac{1}{1 + e^{-t}}$.

(c) As t grows, $e^{-t} \rightarrow 0$, so $x \rightarrow 1$: eventually everyone hears the rumour.

Question 12

[7 marks]

Water drains from a tank so that the depth h metres after t minutes satisfies $\frac{dh}{dt} = -0.2\sqrt{h}$. The depth is 16 m at $t = 0$.

- (a) Show that $h = (4 - 0.1t)^2$. [4]
- (b) Find the time at which the tank is empty. [1]
- (c) Explain why the formula should not be used after that time. [2]

SOLUTION

(a) Separate the variables:

$$\int h^{-\frac{1}{2}} dh = \int -0.2 dt$$
$$2\sqrt{h} = -0.2t + c$$

At $t = 0$, $2 \times 4 = c$, so $c = 8$. Then $\sqrt{h} = 4 - 0.1t$ and $h = (4 - 0.1t)^2$.

(b) $h = 0$ when $4 - 0.1t = 0$, that is after $t = 40$ minutes.

(c) After $t = 40$, $(4 - 0.1t)^2$ grows again, which would mean the tank refilling. The formula came from $\sqrt{h} = 4 - 0.1t$, which needs $4 - 0.1t \geq 0$, so it only describes $0 \leq t \leq 40$.