

YEAR 2 PURE • CHAPTER 7

The Binomial Series

Specification: Edexcel Pure 4.1 — Sequences and series • AQA section D1

KEY FACTS — THIS CHAPTER

- For any rational n and $|x| < 1$: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$
- When n is a positive whole number the series stops. Otherwise it goes on for ever, and is only **valid** for $|x| < 1$.
- For $(1+kx)^n$, replace x by kx throughout. It is valid for $|kx| < 1$, that is $|x| < \frac{1}{|k|}$.
- For $(a+bx)^n$, take out a^n first: $a^n \left(1 + \frac{b}{a}x\right)^n$, valid for $|x| < \left|\frac{a}{b}\right|$.
- To approximate a value, choose x small and inside the interval of validity, then use the first few terms.
- For a fraction, split it into **partial fractions**, expand each one, and add. The result is valid where **both** expansions are.

In Year 1 you expanded $(1+x)^n$ for positive whole numbers n , and the expansion stopped after $n+1$ terms. The same formula works for negative and fractional powers too, but then it never stops. That turns $\frac{1}{1+x}$ and $\sqrt{1+x}$ into series you can use for approximations, as long as x is small enough.

1. The binomial series for any power

KEY RESULT

For any rational number n , and $|x| < 1$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

If n is a positive whole number, a factor $(n-n)$ eventually appears, every later term is 0, and the expansion is the finite one from Year 1. For any other n it is an infinite series.

WORKED EXAMPLE 7.1 • A NEGATIVE POWER

Expand $(1+x)^{-1}$ in ascending powers of x up to the term in x^3 .

With $n = -1$, the coefficients are

$$\begin{aligned} nx &= -x \\ \frac{(-1)(-2)}{2}x^2 &= x^2 \\ \frac{(-1)(-2)(-3)}{6}x^3 &= -x^3 \end{aligned}$$

So $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$, valid for $|x| < 1$. It's the geometric series with first term 1 and ratio $-x$, which converges exactly when $|x| < 1$.

WORKED EXAMPLE 7.2 • A SQUARE ROOT

Expand $\sqrt{1+x}$ up to the term in x^3 .

1. Write it as $(1+x)^{\frac{1}{2}}$, so $n = \frac{1}{2}$.
2. Work out each coefficient:

$$\begin{aligned} \frac{\frac{1}{2}(-\frac{1}{2})}{2} &= -\frac{1}{8} \\ \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6} &= \frac{1}{16} \end{aligned}$$

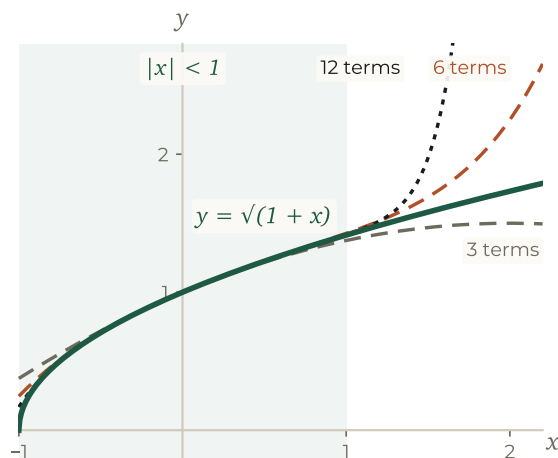
3. So

$$\begin{aligned} \sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 \\ &\quad + \frac{1}{16}x^3 - \dots \end{aligned}$$

valid for $|x| < 1$.

WHY THE TERMS MUST SHRINK

Each term carries a power of x . When $|x| < 1$ the powers shrink, and the terms become small enough for the series to settle on a value. When $|x| > 1$ the powers grow and the series can't settle. Try $x = 2$ in Example 7.1: the series $1 - 2 + 4 - 8 + \dots$ swings further and further, although $(1 + 2)^{-1} = \frac{1}{3}$.



$\sqrt{1+x}$ with its 3-, 6- and 12-term approximations. Inside $|x| < 1$, more terms fit better; outside, more terms make it worse.

2. Expanding $(1 + kx)$ to a power

Replace every x in the formula by kx , brackets and all. The series is valid while $|kx| < 1$.

WORKED EXAMPLE 7.3 • A CUBE IN THE DENOMINATOR

Expand $\frac{1}{(1-2x)^3}$ up to the term in x^3 , and state the values of x for which it is valid.

1. Write it as $(1 - 2x)^{-3}$: $n = -3$, with $-2x$ in place of x .
2. Term by term, keeping the brackets:

$$\begin{aligned}
 & 1 + (-3)(-2x) + \frac{(-3)(-4)}{2}(-2x)^2 \\
 & \quad + \frac{(-3)(-4)(-5)}{6}(-2x)^3 \\
 & = 1 + 6x + 24x^2 + 80x^3 + \dots
 \end{aligned}$$

3. It is valid for $|2x| < 1$, that is $|x| < \frac{1}{2}$.

COMMON MISTAKE

Put the whole of kx in a bracket before you raise it to a power: $(-2x)^2 = 4x^2$ and $(-2x)^3 = -8x^3$. Writing $-2x^2$ gets both the number and the sign wrong.

WORKED EXAMPLE 7.4 • FINDING THE CONSTANTS

In the expansion of $(1 + ax)^n$, the coefficient of x is -6 and the coefficient of x^2 is 27 . Find a and n , and state the values of x for which the expansion is valid.

1. Write down the two coefficients:

$$\begin{aligned} na &= -6 \\ \frac{n(n-1)}{2}a^2 &= 27 \end{aligned}$$

2. Square the first, $n^2a^2 = 36$, and double the second, $n(n-1)a^2 = 54$. Divide the second by the first:

$$\frac{n-1}{n} = \frac{3}{2} \Rightarrow n = -2$$

3. Then $a = \frac{-6}{-2} = 3$. The expansion is of $(1 + 3x)^{-2}$, valid for $|3x| < 1$, that is $|x| < \frac{1}{3}$.

3. Expanding $(a + bx)$ to a power

The formula needs a 1 at the front of the bracket. So take a out first, remembering that it comes out raised to the power n .

KEY RESULT

$$(a + bx)^n = a^n \left(1 + \frac{b}{a}x\right)^n$$

Expand the bracket as before, with $\frac{b}{a}x$ in place of x , then multiply every term by a^n . The result is valid for $\left|\frac{b}{a}x\right| < 1$.

WORKED EXAMPLE 7.5 • TAKING OUT THE 4

Expand $\sqrt{4+x}$ up to the term in x^3 , and state the values of x for which it is valid.

1. Take out $4^{\frac{1}{2}} = 2$:

$$\sqrt{4+x} = 2 \left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$$

2. Use Example 7.2 with $\frac{x}{4}$ in place of x :

$$\begin{aligned} & 2 \left(1 + \frac{x}{4} - \frac{x^2}{128} + \frac{x^3}{1024}\right) \\ &= 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} - \dots \end{aligned}$$

3. It is valid for $\left|\frac{x}{4}\right| < 1$, that is $|x| < 4$.

COMMON MISTAKE

The 4 you take out is raised to the power too. So $\sqrt{4+x} = 2 \left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$, not $4 \left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$, which is twice as big.

4. Approximations

The first few terms of a series make a good approximation when x is small, because the later terms carry high powers of a small number.

WORKED EXAMPLE 7.6 • ESTIMATING A SQUARE ROOT

Use the expansion in Example 7.5 to estimate $\sqrt{4.1}$.

1. $\sqrt{4.1} = \sqrt{4+x}$ with $x = 0.1$, which is well inside $|x| < 4$.
2. Substitute into the four terms:

$$\begin{aligned} & 2 + 0.025 - 0.00015625 \\ & \quad + 0.000001953125 \\ &= 2.024845703\dots \end{aligned}$$

3. The true value is $\sqrt{4.1} = 2.024845673\dots$, so four terms are already correct to 7 decimal places.

When the number isn't of the form $a + x$ with x small, look for a value of x that turns the bracket into something related to it.

WORKED EXAMPLE 7.7 · APPROXIMATING $\sqrt{2}$

Expand $\sqrt{1 - 2x}$ up to the term in x^3 . By substituting $x = 0.01$, find an approximation to $\sqrt{2}$.

1. Use Example 7.2 with $-2x$ in place of x :

$$\begin{aligned} & 1 + \frac{1}{2}(-2x) - \frac{1}{8}(-2x)^2 \\ & \quad + \frac{1}{16}(-2x)^3 \\ & = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \dots \end{aligned}$$

valid for $|x| < \frac{1}{2}$, so $x = 0.01$ is allowed.

2. When $x = 0.01$, the left side is $\sqrt{0.98} = \sqrt{\frac{49 \times 2}{100}} = \frac{7}{10}\sqrt{2}$, and the series gives $1 - 0.01 - 0.00005 - 0.0000005 = 0.9899495$.
3. So

$$\begin{aligned} \sqrt{2} & \approx \frac{10}{7} \times 0.9899495 \\ & = 1.4142136 \text{ (7 d.p.)} \end{aligned}$$

which agrees with $\sqrt{2} = 1.41421356\dots$ to 7 decimal places.

5. Using partial fractions

A fraction with two factors underneath is easiest to expand in pieces: split it into partial fractions, expand each, and add.

WORKED EXAMPLE 7.8 • EXPANDING A FRACTION

Given that $\frac{1+5x}{(1-x)(1+2x)} \equiv \frac{2}{1-x} - \frac{1}{1+2x}$, expand it up to the term in x^3 and state the values of x for which it is valid.

1. Expand each partial fraction up to the term in x^3 :

$$\begin{aligned}\frac{2}{1-x} &= 2 + 2x + 2x^2 + 2x^3 \\ \frac{1}{1+2x} &= 1 - 2x + 4x^2 - 8x^3\end{aligned}$$

2. Subtract:

$$\begin{aligned}\frac{1+5x}{(1-x)(1+2x)} \\ = 1 + 4x - 2x^2 + 10x^3 + \dots\end{aligned}$$

3. The first series is valid for $|x| < 1$ and the second for $|x| < \frac{1}{2}$. Both hold when $|x| < \frac{1}{2}$, so that is where the expansion is valid.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

Expand $(1+x)^{-2}$ in ascending powers of x up to the term in x^3 , and state the values of x for which the expansion is valid.

SOLUTION

With $n = -2$:

$$\begin{aligned} 1 + (-2)x + \frac{(-2)(-3)}{2}x^2 \\ + \frac{(-2)(-3)(-4)}{6}x^3 \\ = 1 - 2x + 3x^2 - 4x^3 + \dots \end{aligned}$$

It is valid for $|x| < 1$.

Question 2

[4 marks]

Expand $\sqrt[3]{1+3x}$ up to the term in x^3 , and state the values of x for which the expansion is valid.

SOLUTION

Write it as $(1+3x)^{\frac{1}{3}}$: $n = \frac{1}{3}$, with $3x$ in place of x .

$$\begin{aligned} 1 + \frac{1}{3}(3x) + \frac{\frac{1}{3}(-\frac{2}{3})}{2}(3x)^2 \\ + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{6}(3x)^3 \\ = 1 + x - x^2 + \frac{5}{3}x^3 - \dots \end{aligned}$$

It is valid for $|3x| < 1$, that is $|x| < \frac{1}{3}$.

Question 3

[2 marks]

State the values of x for which the binomial expansion of each of these is valid.

(a) $(1-4x)^{\frac{1}{2}}$ [1]

(b) $(3+2x)^{-2}$ [1]

SOLUTION

(a) $|4x| < 1$, that is $|x| < \frac{1}{4}$.

(b) $(3+2x)^{-2} = 3^{-2} \left(1 + \frac{2}{3}x\right)^{-2}$, so it is valid for $|\frac{2}{3}x| < 1$, that is $|x| < \frac{3}{2}$.

Question 4

[3 marks]

Expand $\frac{1}{2+x}$ up to the term in x^3 , and state the values of x for which the expansion is valid.

SOLUTION

Take out $2^{-1} = \frac{1}{2}$: $\frac{1}{2+x} = \frac{1}{2} \left(1 + \frac{x}{2}\right)^{-1}$.

$$\begin{aligned} & \frac{1}{2} \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8}\right) \\ &= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots \end{aligned}$$

It is valid for $\left|\frac{x}{2}\right| < 1$, that is $|x| < 2$.

Standard**Question 5**

[6 marks]

- (a) Expand $(1 - x)^{-\frac{1}{2}}$ up to the term in x^3 . [3]
- (b) By substituting $x = 0.1$, find an approximation to $\frac{1}{\sqrt{0.9}}$, giving your answer to 5 decimal places. [1]
- (c) Hence find an approximation to $\sqrt{10}$. [2]

SOLUTION

(a) $n = -\frac{1}{2}$, with $-x$ in place of x :

$$\begin{aligned} & 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-x)^2 \\ & \quad + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}(-x)^3 \\ &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \end{aligned}$$

(b) $1 + 0.05 + 0.00375 + 0.0003125 = 1.0540625$, which is 1.05406 to 5 decimal places.

(c) $\frac{1}{\sqrt{0.9}} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}$, so

$$\begin{aligned} \sqrt{10} &\approx 3 \times 1.0540625 \\ &= 3.16219 \text{ (5 d.p.)} \end{aligned}$$

The true value is 3.16228 to 5 decimal places.

Question 6

[4 marks]

Expand $\frac{1+2x}{(1-x)^2}$ in ascending powers of x up to the term in x^3 , and state the values of x for which the expansion is valid.

SOLUTION

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ (Question 1 with $-x$ in place of x). Multiply by $1+2x$, keeping terms up to x^3 :

$$\begin{aligned} &1 + 2x + 3x^2 + 4x^3 \\ &\quad + 2x + 4x^2 + 6x^3 + \dots \\ &= 1 + 4x + 7x^2 + 10x^3 + \dots \end{aligned}$$

It is valid for $|x| < 1$.

Question 7

[6 marks]

Given that $(1+kx)^n = 1 + 2x - 2x^2 + \dots$, where n is not a positive integer,

- (a) find the values of k and n [4]
- (b) find the coefficient of x^3 [1]
- (c) state the values of x for which the expansion is valid. [1]

SOLUTION

(a) $nk = 2$ and $\frac{n(n-1)}{2}k^2 = -2$. So $n^2k^2 = 4$ and $n(n-1)k^2 = -4$. Dividing,

$$\frac{n-1}{n} = -1 \Rightarrow n = \frac{1}{2}$$

and $k = \frac{2}{n} = 4$.

(b)

$$\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6} \times 4^3 = \frac{1}{16} \times 64 = 4$$

(c) $|4x| < 1$, that is $|x| < \frac{1}{4}$.

Question 8

[8 marks]

$$f(x) = \frac{4 - 5x}{(1 + x)(2 - x)}$$

- (a) Express $f(x)$ in partial fractions. [3]
- (b) Hence find the expansion of $f(x)$ in ascending powers of x up to the term in x^2 . [4]
- (c) State the values of x for which the expansion is valid. [1]

SOLUTION

(a) $4 - 5x \equiv A(2 - x) + B(1 + x)$. Put $x = -1$: $9 = 3A$, so $A = 3$. Put $x = 2$: $-6 = 3B$, so $B = -2$.

$$f(x) = \frac{3}{1 + x} - \frac{2}{2 - x}$$

(b) Expand each part. For the second, take out the 2: $\frac{2}{2 - x} = \left(1 - \frac{x}{2}\right)^{-1}$.

$$\begin{aligned}\frac{3}{1 + x} &= 3 - 3x + 3x^2 - \dots \\ \frac{2}{2 - x} &= 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots\end{aligned}$$

Subtracting, $f(x) = 2 - \frac{7}{2}x + \frac{11}{4}x^2 + \dots$

(c) The first expansion needs $|x| < 1$ and the second $|x| < 2$. Both hold when $|x| < 1$.

Question 9

[6 marks]

- (a) Expand $\sqrt[3]{8 + x}$ up to the term in x^2 . [3]
- (b) Use your expansion to estimate $\sqrt[3]{9}$, to 4 decimal places. [2]
- (c) Explain why the expansion should not be used with $x = 19$ to estimate $\sqrt[3]{27}$. [1]

SOLUTION

(a) Take out $8^{\frac{1}{3}} = 2$:

$$\begin{aligned}2\left(1 + \frac{x}{8}\right)^{\frac{1}{3}} \\ &= 2\left(1 + \frac{x}{24} - \frac{x^2}{576} + \dots\right) \\ &= 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \dots\end{aligned}$$

(b) Put $x = 1$: $2 + \frac{1}{12} - \frac{1}{288} = \frac{599}{288} = 2.0799$ (4 d.p.).

(c) The expansion is only valid for $|\frac{x}{8}| < 1$, that is $|x| < 8$. $x = 19$ is outside this interval, so the series does not converge to $\sqrt[3]{27}$.

Exam style

Question 10

[8 marks]

$$f(x) = \frac{2+x}{\sqrt{4-x}}$$

- (a) Find the binomial expansion of $(4-x)^{-\frac{1}{2}}$ up to the term in x^2 . [4]
 (b) State the values of x for which it is valid. [1]
 (c) Hence find the expansion of $f(x)$ up to the term in x^2 . [3]

SOLUTION

(a) Take out $4^{-\frac{1}{2}} = \frac{1}{2}$:

$$\begin{aligned} & \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}} \\ &= \frac{1}{2} \left(1 + \frac{x}{8} + \frac{3x^2}{128} + \dots\right) \\ &= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \dots \end{aligned}$$

The middle line uses $n = -\frac{1}{2}$ with $-\frac{x}{4}$ in place of x : $(-\frac{1}{2})(-\frac{x}{4}) = \frac{x}{8}$ and $\frac{3}{8}(-\frac{x}{4})^2 = \frac{3x^2}{128}$.

(b) $|\frac{x}{4}| < 1$, that is $|x| < 4$.

(c) Multiply by $2+x$ and keep terms up to x^2 :

$$\begin{aligned} & 1 + \frac{1}{8}x + \frac{3}{128}x^2 + \frac{1}{2}x + \frac{1}{16}x^2 \\ &= 1 + \frac{5}{8}x + \frac{11}{128}x^2 + \dots \end{aligned}$$

Question 11

[9 marks]

$$f(x) = \frac{2+5x+x^2}{(1-x)(1+x)^2}$$

- (a) Express $f(x)$ in the form $\frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$. [4]
 (b) Hence show that, in the expansion of $f(x)$ in ascending powers of x , the coefficient of x^2 is zero, and find the coefficient of x^3 . [4]

(c) State the values of x for which the expansion is valid. [1]

SOLUTION

(a) $2 + 5x + x^2 \equiv A(1+x)^2 + B(1-x)(1+x) + C(1-x).$

- Put $x = 1$: $8 = 4A$, so $A = 2$.
- Put $x = -1$: $-2 = 2C$, so $C = -1$.
- Compare x^2 terms: $1 = A - B$, so $B = 1$.

$$f(x) = \frac{2}{1-x} + \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

(b) Expand each part:

$$\begin{aligned}\frac{2}{1-x} &= 2 + 2x + 2x^2 + 2x^3 \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 \\ -\frac{1}{(1+x)^2} &= -1 + 2x - 3x^2 + 4x^3\end{aligned}$$

The x^2 terms add to $2 + 1 - 3 = 0$, and the x^3 terms to $2 - 1 + 4 = 5$. So $f(x) = 2 + 3x + 5x^3 + \dots$, and the coefficient of x^3 is 5.

(c) All three expansions are valid for $|x| < 1$.

Question 12

[9 marks]

- (a) Expand $\sqrt{1-8x}$ up to the term in x^3 . [3]
- (b) State the values of x for which the expansion is valid. [1]
- (c) By substituting $x = \frac{1}{100}$, show that $\sqrt{23} \approx 4.79584$. [3]
- (d) Find the percentage error in this approximation, to 1 significant figure. [2]

SOLUTION

(a) Use $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$ with $-8x$ in place of x :

$$\begin{aligned}1 - 4x - \frac{1}{8}(64x^2) + \frac{1}{16}(-512x^3) \\ = 1 - 4x - 8x^2 - 32x^3 - \dots\end{aligned}$$

(b) $|8x| < 1$, that is $|x| < \frac{1}{8}$.

(c) When $x = \frac{1}{100}$, the left side is $\sqrt{0.92} = \sqrt{\frac{4 \times 23}{100}} = \frac{1}{5}\sqrt{23}$. The series gives

$$\begin{aligned} 1 - 0.04 - 0.0008 - 0.000032 \\ = 0.959168 \end{aligned}$$

so $\sqrt{23} \approx 5 \times 0.959168 = 4.79584$.

(d) $\sqrt{23} = 4.7958315\dots$, so the percentage error is

$$\begin{aligned} \frac{4.79584 - 4.7958315\dots}{4.7958315\dots} \times 100 \\ = 0.0002\% \end{aligned}$$

to 1 significant figure.