

YEAR 2 PURE • CHAPTER 5

Sequences and Arithmetic Series

Specification: Edexcel Pure 4.2–4.4 — Sequences and series • AQA section D2, D3, D4

KEY FACTS — THIS CHAPTER

- A sequence can be given by a formula for its n th term, $u_n = f(n)$, or by a **recurrence relation** $u_{n+1} = f(u_n)$ with a first term.
- **Increasing:** $u_{n+1} > u_n$ for every n . **Decreasing:** $u_{n+1} < u_n$. **Periodic:** the terms repeat, $u_{n+k} = u_n$, and the smallest such k is the **order**.
- **Sigma notation:** $\sum_{r=1}^n u_r = u_1 + u_2 + \cdots + u_n$.
- An **arithmetic** sequence adds the same **common difference** d each time: $u_n = a + (n-1)d$.
- The sum of the first n terms is $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + l)$, where l is the last term.
- In particular $\sum_{r=1}^n r = \frac{n(n+1)}{2}$.

A sequence is a list of numbers made by a rule. Some rules say how to find any term directly; others say how to get from one term to the next. This chapter covers both, the notation for adding terms up, and the first family with a formula for its sum: the arithmetic series.

1. Two ways to define a sequence

KEY RESULT

- An **n th-term formula** gives each term from its position: $u_n = f(n)$. For $u_n = n^2 + 1$ the terms are 2, 5, 10, 17, ...
- A **recurrence relation** gives each term from the one before: $u_{n+1} = f(u_n)$, together with a starting value such as u_1 .

WORKED EXAMPLE 5.1 • TERMS FROM A RECURRENCE RELATION

A sequence is defined by $u_{n+1} = 3u_n - 2$, $u_1 = 2$. Find the next three terms.

Apply the rule to each term in turn:

$$u_2 = 3(2) - 2 = 4$$

$$u_3 = 3(4) - 2 = 10$$

$$u_4 = 3(10) - 2 = 28$$

WORKED EXAMPLE 5.2 • FINDING A CONSTANT

A sequence is defined by $u_{n+1} = ku_n + 3$, $u_1 = 2$, where k is a constant. Given that $u_3 = 23$, find the possible values of k .

1. Work forward from u_1 :

$$u_2 = 2k + 3$$

$$u_3 = k(2k + 3) + 3$$

2. Set $u_3 = 23$ and solve:

$$2k^2 + 3k - 20 = 0$$

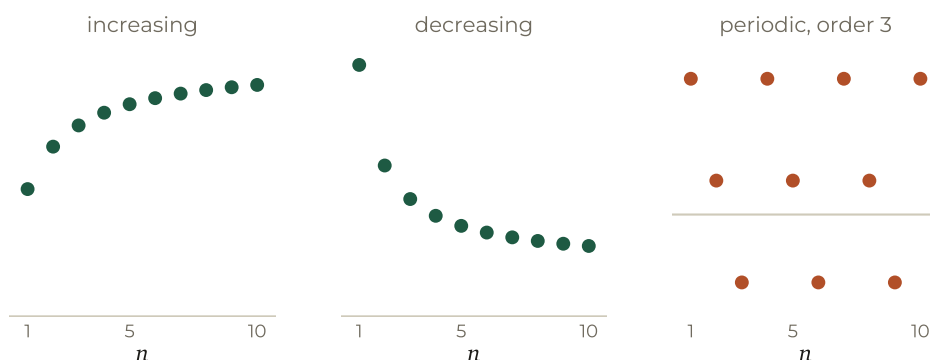
$$(2k - 5)(k + 4) = 0$$

3. So $k = \frac{5}{2}$ or $k = -4$.

2. Increasing, decreasing and periodic

KEY RESULT

- A sequence is **increasing** if $u_{n+1} > u_n$ for every n , and **decreasing** if $u_{n+1} < u_n$ for every n .
- It is **periodic** if the terms repeat in a cycle: $u_{n+k} = u_n$ for every n . The smallest such k is the **order** of the sequence.



Increasing, decreasing, and periodic with order 3.

To show a sequence is increasing, show that $u_{n+1} - u_n$ is positive for every n , not just for the first few.

WORKED EXAMPLE 5.3 • PROVING A SEQUENCE IS INCREASING

Show that the sequence $u_n = \frac{3n}{n+1}$ is increasing.

1. Subtract consecutive terms over a common denominator:

$$\begin{aligned}
 & u_{n+1} - u_n \\
 &= \frac{3(n+1)}{n+2} - \frac{3n}{n+1} \\
 &= \frac{3(n+1)^2 - 3n(n+2)}{(n+1)(n+2)} \\
 &= \frac{3}{(n+1)(n+2)}
 \end{aligned}$$

2. For $n \geq 1$ the top and bottom are both positive, so $u_{n+1} - u_n > 0$ and the sequence is increasing.

WORKED EXAMPLE 5.4 · A PERIODIC SEQUENCE

A sequence is defined by $u_{n+1} = 1 - \frac{1}{u_n}$, $u_1 = 2$. Show that it is periodic, state its order, and find u_{100} .

1. The first few terms:

$$u_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$u_3 = 1 - 2 = -1$$

$$u_4 = 1 - (-1) = 2$$

2. $u_4 = u_1$, and each term depends only on the one before, so the terms repeat $2, \frac{1}{2}, -1$ for ever.
The sequence is periodic with order 3.

3. $100 = 3 \times 33 + 1$, so u_{100} is at the same point in the cycle as u_1 : $u_{100} = 2$.

3. Sigma notation

The Greek capital sigma means “add up”. Below and above it go the first and last values of the counter r .

KEY RESULT

$$\sum_{r=1}^n u_r = u_1 + u_2 + u_3 + \cdots + u_n$$

For example,

$$\begin{aligned} \sum_{r=1}^5 (2r + 1) &= 3 + 5 + 7 + 9 + 11 \\ &= 35 \end{aligned}$$

COMMON MISTAKE

Count the terms carefully. $\sum_{r=5}^{20}$ has $20 - 5 + 1 = 16$ terms, not 15: both ends are included.

4. Arithmetic sequences

An **arithmetic** sequence goes up or down by the same amount every time, like 5, 8, 11, 14, ... That amount is the **common difference**.

KEY RESULT

With first term a and common difference d , the n th term is

$$u_n = a + (n - 1)d$$

The n th term needs $n - 1$ steps of d from the first term.

WORKED EXAMPLE 5.5 · TWO TERMS FIX THE SEQUENCE

The 5th term of an arithmetic sequence is 17 and the 12th term is 45. Find the first term and the common difference, and how many terms are less than 200.

1. Write both terms using the formula:

$$a + 4d = 17$$

$$a + 11d = 45$$

2. Subtracting, $7d = 28$, so $d = 4$ and $a = 1$.
3. The n th term is $1 + 4(n - 1) = 4n - 3$. For $4n - 3 < 200$ we need $n < 50.75$, so 50 terms are less than 200.

5. Arithmetic series

Adding up the terms of an arithmetic sequence gives an **arithmetic series**. There's a neat trick for the sum, said to have been found by the mathematician Gauss as a schoolboy.

THE SUM OF AN ARITHMETIC SERIES

The sum of the first n terms, with first term a , common difference d and last term l , is

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + l)$$

In particular, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

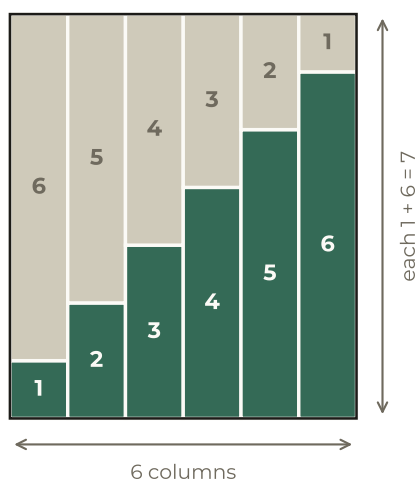
PROOF

Write the sum forwards, then backwards underneath:

$$S_n = a + (a + d) + \cdots + l$$

$$S_n = l + (l - d) + \cdots + a$$

Add the two lines pair by pair. Every pair adds to $a + l$, and there are n pairs, so $2S_n = n(a + l)$. Since $l = a + (n-1)d$, this gives $S_n = \frac{n}{2}(2a + (n-1)d)$.



Two copies of $1 + 2 + \dots + 6$ fill a 6×7 rectangle, so the sum is half of 42.

WORKED EXAMPLE 5.6 • A SUM WITH A KNOWN LAST TERM

Find the sum $1 + 5 + 9 + \cdots + 197$.

- Here $a = 1$ and $d = 4$. The last term is the n th, where $1 + 4(n-1) = 197$, so $n = 50$.
- So

$$S_{50} = \frac{50}{2}(1 + 197) = 4950$$

WORKED EXAMPLE 5.7 • WHEN DOES THE SUM PASS A TARGET?

Find the smallest number of terms of $3 + 5 + 7 + \dots$ that add up to more than 1000.

1. With $a = 3$ and $d = 2$:

$$\begin{aligned} S_n &= \frac{n}{2}(6 + 2(n - 1)) \\ &= n(n + 2) \end{aligned}$$

2. Solve $n^2 + 2n - 1000 = 0$: the positive root is $n = -1 + \sqrt{1001} = 30.6$ (3 s.f.).

3. n must be a whole number, so $n = 31$. Check: $S_{30} = 960$ and $S_{31} = 1023$.

WORKED EXAMPLE 5.8 • A SUM IN SIGMA NOTATION

Evaluate $\sum_{r=1}^{20} (3r + 2)$.

The terms are 5, 8, 11, ..., 62: an arithmetic series with 20 terms, first term 5 and last term 62. So

$$\sum_{r=1}^{20} (3r + 2) = \frac{20}{2}(5 + 62) = 670$$

SERIES IN CONTEXT

A salary starts at £24 000 and rises by £800 each year. Over 10 years the total earned is an arithmetic series with $a = 24\,000$ and $d = 800$: $S_{10} = 5(48\,000 + 9 \times 800) = £276\,000$.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

A sequence has n th term $u_n = n^2 - 3n$.

- (a) Find u_5 . [1]
(b) Which term of the sequence is equal to 40? [2]

SOLUTION

- (a) $u_5 = 25 - 15 = 10$.
(b) $n^2 - 3n = 40$ gives $n^2 - 3n - 40 = 0$, so $(n - 8)(n + 5) = 0$. n is a positive integer, so it is the 8th term.

Question 2

[3 marks]

A sequence is defined by $u_{n+1} = 2u_n - 1$, $u_1 = 3$.

- (a) Find u_2 , u_3 and u_4 . [2]

- (b) Find $\sum_{r=1}^4 u_r$. [1]

SOLUTION

- (a) $u_2 = 2(3) - 1 = 5$, $u_3 = 2(5) - 1 = 9$ and $u_4 = 2(9) - 1 = 17$.
(b) $3 + 5 + 9 + 17 = 34$.

Question 3

[3 marks]

An arithmetic sequence begins 7, 11, 15, ...

- (a) Find the 20th term. [1]
(b) Find the sum of the first 20 terms. [2]

SOLUTION

- (a) $a = 7$ and $d = 4$, so $u_{20} = 7 + 19 \times 4 = 83$.
(b) $S_{20} = \frac{20}{2}(7 + 83) = 900$.

Question 4

[2 marks]

Evaluate $\sum_{r=1}^{10} (4r - 1)$.

SOLUTION

The terms are 3, 7, 11, ..., 39: an arithmetic series with 10 terms. So

$$\sum_{r=1}^{10} (4r - 1) = \frac{10}{2} (3 + 39) = 210$$

Standard**Question 5**

[4 marks]

The 3rd term of an arithmetic sequence is 13 and the 10th term is 48. Find the first term, the common difference, and the sum of the first 25 terms.

SOLUTION

$a + 2d = 13$ and $a + 9d = 48$. Subtracting, $7d = 35$, so $d = 5$ and $a = 3$.

$$\begin{aligned} S_{25} &= \frac{25}{2} (2 \times 3 + 24 \times 5) \\ &= \frac{25}{2} \times 126 = 1575 \end{aligned}$$

Question 6

[4 marks]

A sequence is defined by $u_{n+1} = ku_n - 4$, $u_1 = 3$, where k is a positive constant. Given that $u_3 = 11$, find the value of k .

SOLUTION

$u_2 = 3k - 4$, so $u_3 = k(3k - 4) - 4 = 3k^2 - 4k - 4$. Setting this equal to 11:

$$\begin{aligned} 3k^2 - 4k - 15 &= 0 \\ (3k + 5)(k - 3) &= 0 \end{aligned}$$

k is positive, so $k = 3$.

Question 7

[4 marks]

Show that the sequence $u_n = \frac{2n+1}{n+1}$ is increasing.

SOLUTION

$$u_{n+1} - u_n = \frac{2n+3}{n+2} - \frac{2n+1}{n+1}$$

Over the common denominator $(n+1)(n+2)$, the numerator is

$$\begin{aligned} & (2n+3)(n+1) - (2n+1)(n+2) \\ &= 2n^2 + 5n + 3 - 2n^2 - 5n - 2 \\ &= 1 \end{aligned}$$

So $u_{n+1} - u_n = \frac{1}{(n+1)(n+2)}$, which is positive for $n \geq 1$: $u_{n+1} > u_n$ and the sequence is increasing.

Question 8

[4 marks]

A sequence is defined by $u_{n+1} = \frac{1}{1-u_n}$, $u_1 = 2$.

(a) Show that the sequence is periodic and state its order. [2]

(b) Find $\sum_{r=1}^{30} u_r$. [2]

SOLUTION

(a) $u_2 = \frac{1}{1-2} = -1$, $u_3 = \frac{1}{1-\frac{1}{2}} = 2 = u_1$. The terms repeat 2, -1, $\frac{1}{2}$, so the sequence is periodic with order 3.

(b) Each cycle adds to $2 - 1 + \frac{1}{2} = \frac{3}{2}$, and 30 terms make 10 cycles. The sum is $10 \times \frac{3}{2} = 15$.

Question 9

[4 marks]

Find the smallest number of terms of the series $5 + 8 + 11 + \dots$ whose sum is greater than 500.

SOLUTION

With $a = 5$ and $d = 3$, $S_n = \frac{n}{2}(3n+7)$. Solve $3n^2 + 7n - 1000 = 0$: the positive root is

$$n = \frac{-7 + \sqrt{12049}}{6} = 17.1 \text{ (3 s.f.)}$$

So $n = 18$. Check: $S_{17} = 493$ and $S_{18} = 549$.

Exam style**Question 10**

[7 marks]

- (a) Prove that the sum of the first n terms of an arithmetic series with first term a and common difference d is $\frac{n}{2}(2a + (n - 1)d)$. [4]
- (b) Find the sum of all the multiples of 7 between 100 and 500. [3]

SOLUTION

(a) Write the sum forwards and backwards, where $l = a + (n - 1)d$ is the last term:

$$S_n = a + (a + d) + \cdots + (l - d) + l$$

$$S_n = l + (l - d) + \cdots + (a + d) + a$$

Adding term by term, each of the n pairs sums to $a + l$, so

$$2S_n = n(a + l) = n(2a + (n - 1)d)$$

and dividing by 2 gives the result.

(b) The multiples of 7 from $105 = 15 \times 7$ to $497 = 71 \times 7$ form an arithmetic series with $71 - 15 + 1 = 57$ terms:

$$\begin{aligned} S &= \frac{57}{2}(105 + 497) \\ &= 57 \times 301 = 17\,157 \end{aligned}$$

Question 11

[6 marks]

Ali saves £50 in the first month. Each month after that, he saves £5 more than the month before.

- (a) How much does he save in the 12th month? [1]
- (b) How much has he saved in total after 12 months? [2]
- (c) In which month does his total savings first go over £3000? [3]

SOLUTION

(a) $50 + 11 \times 5 = £105$.

(b) $S_{12} = \frac{12}{2}(50 + 105) = £930$.

(c) $S_n = \frac{n}{2}(100 + 5(n-1)) = \frac{n(5n+95)}{2}$. Setting $S_n = 3000$ gives $n^2 + 19n - 1200 = 0$, with positive root

$$n = \frac{-19 + \sqrt{5161}}{2} = 26.4 \text{ (3 s.f.)}$$

Check: $S_{26} = £2925$ and $S_{27} = £3105$. So his total first goes over £3000 in the 27th month.

Question 12

[6 marks]

The sum of the first 10 terms of an arithmetic series is 175, and the sum of the first 20 terms is 650.

- (a) Find the first term and the common difference. [4]
- (b) Find the sum of the 21st to the 30th terms. [2]

SOLUTION

(a) $S_{10} = 5(2a + 9d) = 175$, so $2a + 9d = 35$.

$S_{20} = 10(2a + 19d) = 650$, so $2a + 19d = 65$.

Subtracting, $10d = 30$, so $d = 3$, and then $2a = 35 - 27 = 8$, so $a = 4$.

(b) The 21st to 30th terms are $S_{30} - S_{20}$:

$$\begin{aligned} S_{30} &= 15(8 + 87) = 1425 \\ S_{30} - S_{20} &= 1425 - 650 = 775 \end{aligned}$$