

YEAR 2 PURE · CHAPTER 10

Addition and Double-Angle Formulae

Specification: Edexcel Pure 5.6–5.9 — Trigonometry · AQA section E6, E7, E8, E9

KEY FACTS — THIS CHAPTER

- $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$ and $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$.
- $\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.
- $\sin 2A \equiv 2 \sin A \cos A$, and $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$.
- $\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$.
- $a \cos \theta + b \sin \theta \equiv R \cos(\theta - \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$. Its largest value is R and its smallest is $-R$.
- To solve $a \cos \theta + b \sin \theta = c$, write the left side in this form first.

$\sin(A + B)$ is not $\sin A + \sin B$: try $A = B = \frac{\pi}{2}$, which gives 0 on the left and 2 on the right. This chapter finds what it really is. The formulae that come out of it turn sums of sines and cosines into single waves, which is how a wave's height and timing are found in models.

1. The addition formulae

KEY RESULT

$$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$$

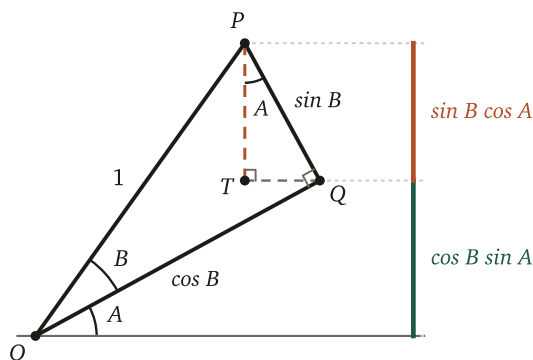
$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Watch the signs in the cosine formulae: a + inside the bracket gives a – between the terms.



The height of P, which is $\sin(A + B)$, split into two pieces.

A GEOMETRIC PROOF

In the figure, $OP = 1$ makes angle $A + B$ with the horizontal, so P is at height $\sin(A + B)$. The right-angled triangle OQP has angle B at O , so $OQ = \cos B$ and $QP = \sin B$.

- Q is at height $OQ \sin A = \cos B \sin A$.
- QP makes angle A with the vertical, so P is a further $QP \cos A = \sin B \cos A$ above Q .

Adding the two pieces, $\sin(A + B) = \sin A \cos B + \cos A \sin B$. The horizontal distances give $\cos(A + B) = \cos A \cos B - \sin A \sin B$ in the same way. Replacing B with $-B$ gives the two minus formulae, and dividing $\sin(A + B)$ by $\cos(A + B)$ gives the one for $\tan$.

WORKED EXAMPLE 10.1 · AN EXACT VALUE

Find the exact value of $\cos \frac{\pi}{12}$.

1. Write $\frac{\pi}{12}$ as a difference of angles you know: $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.
2. Use $\cos(A - B)$:

$$\begin{aligned}
 \cos \frac{\pi}{12} &= \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\
 &= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} \\
 &= \frac{\sqrt{2} + \sqrt{6}}{4}
 \end{aligned}$$

WORKED EXAMPLE 10.2 · TWO ANGLES IN DIFFERENT QUADRANTS

A is acute with $\sin A = \frac{3}{5}$, and B is obtuse with $\cos B = -\frac{5}{13}$. Find the exact value of $\sin(A + B)$.

- Find the missing ratios, with the right signs. A is acute, so $\cos A = \frac{4}{5}$. B is obtuse, where sine is positive, so $\sin B = \frac{12}{13}$.
- Substitute:

$$\begin{aligned}\sin(A + B) &= \frac{3}{5} \times \left(-\frac{5}{13}\right) + \frac{4}{5} \times \frac{12}{13} \\ &= \frac{-15+48}{65} = \frac{33}{65}\end{aligned}$$

WORKED EXAMPLE 10.3 · UNDOING THE FORMULA

Given that $\tan\left(x + \frac{\pi}{4}\right) = 3$, find the exact value of $\tan x$.

$\tan \frac{\pi}{4} = 1$, so the formula gives

$$\begin{aligned}\frac{\tan x + 1}{1 - \tan x} &= 3 \\ \tan x + 1 &= 3 - 3 \tan x \\ \tan x &= \frac{1}{2}\end{aligned}$$

2. Double-angle formulae

Put $B = A$ in the addition formulae and they become the double-angle formulae. Using $\sin^2 A + \cos^2 A \equiv 1$ turns the cosine one into three versions: choose the one that leaves only the function you want.

KEY RESULT

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A$$

$$\equiv 2 \cos^2 A - 1$$

$$\equiv 1 - 2 \sin^2 A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

Rearranging the cosine versions gives $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$ and $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$. With $A = \frac{x}{2}$ these are the half-angle formulae, and they are what you'll use to integrate $\sin^2 x$ and $\cos^2 x$ in Chapter 15.

WORKED EXAMPLE 10.4 · DOUBLE ANGLES FROM A SINGLE RATIO

Given that $\cos x = \frac{1}{3}$ and x is acute, find the exact values of $\sin 2x$ and $\cos 2x$.

1. $\sin x = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$, positive because x is acute.

2. Then

$$\sin 2x = 2 \times \frac{2\sqrt{2}}{3} \times \frac{1}{3} = \frac{4\sqrt{2}}{9}$$

$$\cos 2x = 2 \times \frac{1}{9} - 1 = -\frac{7}{9}$$

WORKED EXAMPLE 10.5 · SOLVING WITH A DOUBLE ANGLE

Solve $\sin 2x = \sin x$ for $0 \leq x \leq 2\pi$.

1. Expand and factorise. Don't divide by $\sin x$, which would lose solutions:

$$2 \sin x \cos x - \sin x = 0$$

$$\sin x (2 \cos x - 1) = 0$$

2. $\sin x = 0$ gives $x = 0, \pi, 2\pi$. $\cos x = \frac{1}{2}$ gives $x = \frac{\pi}{3}, \frac{5\pi}{3}$.

So $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.

WORKED EXAMPLE 10.6 · CHOOSING THE RIGHT VERSION

Solve $\cos 2x = 5 \cos x - 3$ for $0 \leq x < 2\pi$.

1. The right side has $\cos x$, so use $\cos 2x = 2 \cos^2 x - 1$:

$$2 \cos^2 x - 1 = 5 \cos x - 3$$

$$2 \cos^2 x - 5 \cos x + 2 = 0$$

$$(2 \cos x - 1)(\cos x - 2) = 0$$

2. $\cos x = 2$ is impossible, so $\cos x = \frac{1}{2}$ and $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

WORKED EXAMPLE 10.7 · PROVING AN IDENTITY

Prove that $\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan \theta$.

Use $\cos 2\theta \equiv 1 - 2 \sin^2 \theta$, which makes the 1s cancel:

$$\begin{aligned} \frac{1 - \cos 2\theta}{\sin 2\theta} &\equiv \frac{2 \sin^2 \theta}{2 \sin \theta \cos \theta} \\ &\equiv \frac{\sin \theta}{\cos \theta} \equiv \tan \theta \end{aligned}$$

3. The form $R \cos(\theta - \alpha)$

Adding a cosine wave and a sine wave with the same period gives another wave with that period: it's just taller and shifted along. Writing it as a single cosine tells you its height and where its peaks are.

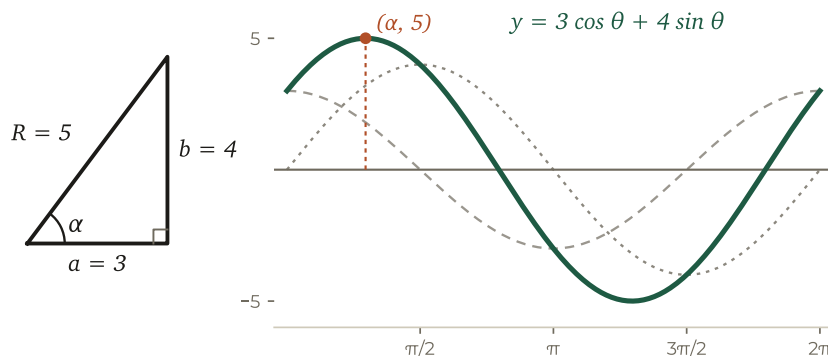
KEY RESULT

For positive a and b ,

$$a \cos \theta + b \sin \theta \equiv R \cos(\theta - \alpha)$$

where $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$, with $0 < \alpha < \frac{\pi}{2}$.

The same method gives the other forms: $R \cos(\theta + \alpha)$ for $a \cos \theta - b \sin \theta$, and $R \sin(\theta \pm \alpha)$ for $a \sin \theta \pm b \cos \theta$.



$3 \cos \theta$ (dashed) plus $4 \sin \theta$ (dotted) is one wave of height $R = 5$, peaking at $\theta = \alpha$. The triangle gives R and α .

To see why, expand the right side:

$$\begin{aligned} R \cos(\theta - \alpha) &= R \cos \alpha \cos \theta \\ &\quad + R \sin \alpha \sin \theta \end{aligned}$$

Matching the coefficients with $a \cos \theta + b \sin \theta$ needs

$$R \cos \alpha = a \quad \text{and} \quad R \sin \alpha = b$$

Squaring and adding gives $R^2 = a^2 + b^2$. Dividing gives $\tan \alpha = \frac{b}{a}$.

WORKED EXAMPLE 10.8 · WRITING IT AS ONE WAVE

Express $3 \cos \theta + 4 \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. State its maximum value and the smallest positive value of θ where it occurs.

1. $R \cos \alpha = 3$ and $R \sin \alpha = 4$, so

$$\begin{aligned} R &= \sqrt{3^2 + 4^2} = 5 \\ \alpha &= \tan^{-1} \frac{4}{3} = 0.9272 \dots \end{aligned}$$

So $3 \cos \theta + 4 \sin \theta = 5 \cos(\theta - 0.927)$, to 3 s.f.

2. The largest value of a cosine is 1, so the maximum is 5, when $\theta - \alpha = 0$, that is $\theta = 0.927$ (3 s.f.).

WORKED EXAMPLE 10.9 · SOLVING WITH THE R FORM

Solve $3 \cos \theta + 4 \sin \theta = 2$ for $0 \leq \theta < 2\pi$, giving your answers to 3 significant figures.

1. From Example 10.8, $5 \cos(\theta - \alpha) = 2$, so $\cos(\theta - \alpha) = 0.4$, where $\alpha = 0.9272 \dots$
2. As θ runs from 0 to 2π , $\theta - \alpha$ runs from -0.927 to 5.356 . In that interval,

$$\theta - \alpha = 1.1592 \dots \text{ or } 5.1239 \dots$$

(-1.159 is too small to be in the interval.)

3. Add α : $\theta = 2.09$ or $\theta = 6.05$ (3 s.f.).

COMMON MISTAKE

Find α from $\tan \alpha = \frac{b}{a}$, not $\frac{a}{b}$: the $\sin \alpha$ coefficient goes on top. Then check by expanding: $5 \cos(\theta - 0.927)$ should give back $3 \cos \theta + 4 \sin \theta$.

4. Modelling with the R form**WORKED EXAMPLE 10.10 · TEMPERATURE IN A GREENHOUSE**

The temperature in a greenhouse, T °C, t hours after midnight is modelled by $T = 20 - 3 \cos\left(\frac{\pi t}{12}\right) - 4 \sin\left(\frac{\pi t}{12}\right)$. Find the highest temperature and the time of day at which it happens.

1. Write $\theta = \frac{\pi t}{12}$. By Example 10.8, $3 \cos \theta + 4 \sin \theta = 5 \cos(\theta - \alpha)$ with $\alpha = 0.9272 \dots$, so $T = 20 - 5 \cos(\theta - \alpha)$.
2. T is largest when the cosine is -1 , so the highest temperature is $20 + 5 = 25$ °C.
3. That happens when $\theta - \alpha = \pi$:

$$\frac{\pi t}{12} = \pi + 0.9272 \dots$$

$$t = 15.542 \text{ (3 d.p.)}$$

0.542 hours is 33 minutes to the nearest minute, so the greenhouse is warmest at about 3:33 pm.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[3 marks]

Find the exact value of $\sin \frac{\pi}{12}$.

SOLUTION

$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$, so use $\sin(A - B)$:

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Question 2

[3 marks]

Write each of these as a single trigonometric function or an exact value.

(a) $\sin 3x \cos x + \cos 3x \sin x$ [1]

(b) $\cos^2 2x - \sin^2 2x$ [1]

(c) $2 \sin \frac{\pi}{8} \cos \frac{\pi}{8}$ [1]

SOLUTION

(a) $\sin(3x + x) = \sin 4x$.

(b) $\cos(2 \times 2x) = \cos 4x$.

(c) $\sin \left(2 \times \frac{\pi}{8}\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Question 3

[3 marks]

A and B are acute angles with $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$. Find $\tan(A + B)$, and hence show that $A + B = \frac{\pi}{4}$.

SOLUTION

$$\tan(A + B) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

A and B are acute, so $0 < A + B < \pi$, and the only angle there with tangent 1 is $\frac{\pi}{4}$.

Question 4

[2 marks]

Given that $\sin x = \frac{1}{4}$, find the exact value of $\cos 2x$.

SOLUTION

Use the version with sin only: $\cos 2x = 1 - 2 \sin^2 x = 1 - \frac{2}{16} = \frac{7}{8}$.

Standard**Question 5**

[4 marks]

Solve $\cos 2x = \cos x$ for $0 \leq x \leq 2\pi$.

SOLUTION

Use $\cos 2x = 2 \cos^2 x - 1$:

$$\begin{aligned} 2 \cos^2 x - \cos x - 1 &= 0 \\ (2 \cos x + 1)(\cos x - 1) &= 0 \end{aligned}$$

$\cos x = 1$ gives $x = 0$ or 2π , and $\cos x = -\frac{1}{2}$ gives $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. So $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$.

Question 6

[4 marks]

Prove that $\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta$.

SOLUTION

$$\begin{aligned}
 \tan \theta + \cot \theta &\equiv \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\
 &\equiv \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \\
 &\equiv \frac{1}{\frac{1}{2} \sin 2\theta} \equiv 2 \operatorname{cosec} 2\theta
 \end{aligned}$$

Question 7

[6 marks]

(a) Express $\sqrt{3} \cos \theta + \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving exact values of R and α . [3]

(b) Hence solve $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$ for $0 \leq \theta < 2\pi$. [3]

SOLUTION

(a) $R \cos \alpha = \sqrt{3}$ and $R \sin \alpha = 1$, so $R = \sqrt{3+1} = 2$ and $\tan \alpha = \frac{1}{\sqrt{3}}$, giving $\alpha = \frac{\pi}{6}$. So the expression is $2 \cos(\theta - \frac{\pi}{6})$.

(b) $\cos(\theta - \frac{\pi}{6}) = \frac{\sqrt{2}}{2}$. As θ runs from 0 to 2π , $\theta - \frac{\pi}{6}$ runs from $-\frac{\pi}{6}$ to $\frac{11\pi}{6}$, and in that interval

$$\theta - \frac{\pi}{6} = \frac{\pi}{4} \text{ or } \frac{7\pi}{4}$$

So $\theta = \frac{5\pi}{12}$ or $\frac{23\pi}{12}$.

Question 8

[4 marks]

A is obtuse with $\cos A = -\frac{4}{5}$, and B is acute with $\sin B = \frac{12}{13}$. Find the exact value of $\cos(A - B)$.

SOLUTION

A is obtuse, so $\sin A = \frac{3}{5}$ (positive). B is acute, so $\cos B = \frac{5}{13}$. Then

$$\begin{aligned}
 \cos(A - B) &= -\frac{4}{5} \times \frac{5}{13} + \frac{3}{5} \times \frac{12}{13} \\
 &= \frac{-20+36}{65} = \frac{16}{65}
 \end{aligned}$$

Question 9

[5 marks]

- (a) Show that $\cos 3x \equiv 4 \cos^3 x - 3 \cos x$. [4]
- (b) Hence find the exact value of $\cos 3x$ when $\cos x = \frac{1}{3}$. [1]

SOLUTION

- (a) Write $3x$ as $2x + x$:

$$\begin{aligned}
 \cos 3x & \\
 &\equiv \cos 2x \cos x - \sin 2x \sin x \\
 &\equiv (2 \cos^2 x - 1) \cos x - 2 \sin^2 x \cos x \\
 &\equiv 2 \cos^3 x - \cos x \\
 &\quad - 2(1 - \cos^2 x) \cos x \\
 &\equiv 4 \cos^3 x - 3 \cos x
 \end{aligned}$$

(b) $4 \times \frac{1}{27} - 3 \times \frac{1}{3} = \frac{4}{27} - 1 = -\frac{23}{27}$.

Exam style**Question 10**

[8 marks]

- (a) Express $5 \sin \theta - 12 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give α to 4 decimal places. [3]
- (b) State the maximum value of $5 \sin \theta - 12 \cos \theta$, and the smallest positive value of θ at which it occurs. [2]
- (c) Solve $5 \sin \theta - 12 \cos \theta = 6.5$ for $0 \leq \theta < 2\pi$, to 3 significant figures. [3]

SOLUTION

(a) $R \sin(\theta - \alpha) = R \cos \alpha \sin \theta - R \sin \alpha \cos \theta$, so $R \cos \alpha = 5$ and $R \sin \alpha = 12$. Then $R = \sqrt{25 + 144} = 13$ and $\alpha = \tan^{-1} \frac{12}{5} = 1.1760$ (4 d.p.).

(b) The maximum is 13, when $\theta - \alpha = \frac{\pi}{2}$, that is $\theta = \frac{\pi}{2} + 1.1760 = 2.75$ (3 s.f.).

(c) $\sin(\theta - \alpha) = \frac{6.5}{13} = \frac{1}{2}$. As θ runs from 0 to 2π , $\theta - \alpha$ runs from -1.176 to 5.107 , and in that interval $\theta - \alpha = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. So

$$\theta = 1.1760 + 0.5236 = 1.70$$

$$\theta = 1.1760 + 2.6180 = 3.79$$

to 3 significant figures.

Question 11

[8 marks]

The depth of water in a harbour, D metres, t hours after midnight is modelled by

$$D = 5 + 1.5 \cos\left(\frac{\pi t}{6}\right) + 2 \sin\left(\frac{\pi t}{6}\right)$$

- (a) Write D in the form $5 + R \cos\left(\frac{\pi t}{6} - \alpha\right)$, giving R exactly and α to 4 decimal places. [3]
- (b) Find the greatest depth, and the first time after midnight when it occurs, to the nearest minute. [3]
- (c) State the least depth. [1]
- (d) Give one reason why the model may not be realistic. [1]

SOLUTION

- (a) $R \cos \alpha = 1.5$ and $R \sin \alpha = 2$, so $R = \sqrt{2.25 + 4} = 2.5$ and $\alpha = \tan^{-1} \frac{4}{3} = 0.9273$ (4 d.p.). So $D = 5 + 2.5 \cos\left(\frac{\pi t}{6} - 0.9273\right)$.
- (b) The greatest depth is $5 + 2.5 = 7.5$ m, when the cosine is 1, so $\frac{\pi t}{6} = 0.9272 \dots$ and $t = 1.7710 \dots$. Then 0.7710 hours is 46 minutes, so this is at about 1:46 am.
- (c) $5 - 2.5 = 2.5$ m.
- (d) For example, real tides don't repeat exactly every 12 hours, and the depth also depends on the weather.

Question 12

[6 marks]

- (a) Prove that $\frac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan \theta$. [3]
- (b) Hence solve $\frac{\sin 4x}{1 + \cos 4x} = \sqrt{3}$ for $0 \leq x \leq \pi$. [3]

SOLUTION

- (a) Use $\cos 2\theta \equiv 2 \cos^2 \theta - 1$, which makes the 1s cancel:

$$\frac{\sin 2\theta}{1 + \cos 2\theta} \equiv \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta} \equiv \tan \theta$$

- (b) With $\theta = 2x$, the equation is $\tan 2x = \sqrt{3}$. As x runs from 0 to π , $2x$ runs from 0 to 2π , so $2x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$, and $x = \frac{\pi}{6}$ or $\frac{2\pi}{3}$. At both, $1 + \cos 4x = \frac{1}{2} \neq 0$, so the original fraction is defined.