

YEAR 2 MECHANICS · CHAPTER 3

Projectiles

Specification: Edexcel Paper 3 §7.5 — Projectiles · AQA section Q5

KEY FACTS — THIS CHAPTER

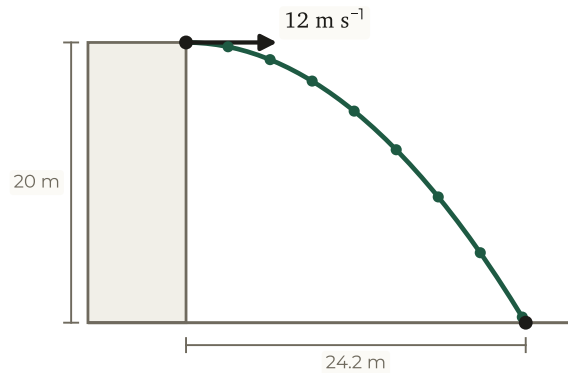
- Treat the **horizontal** and **vertical** motion separately. Ignoring air resistance, the only acceleration is g vertically downwards.
- Projected at speed U at angle α above the horizontal: horizontal component $U \cos \alpha$, which never changes; vertical component $U \sin \alpha$, which changes at g .
- The **time** links the two directions: find it from one, use it in the other.
- On level ground: time of flight $T = \frac{2U \sin \alpha}{g}$, greatest height $H = \frac{U^2 \sin^2 \alpha}{2g}$, range $R = \frac{U^2 \sin 2\alpha}{g}$, greatest at $\alpha = 45^\circ$.
- The path is a parabola: $y = x \tan \alpha - \frac{gx^2}{2U^2 \cos^2 \alpha}$.
- The velocity at any time has components $U \cos \alpha$ and $U \sin \alpha - gt$: use Pythagoras for the speed and $\tan^{-1}$ for the direction.

A thrown ball moves forwards and up and down at the same time. The trick is that these two motions don't interfere: horizontally nothing acts on the ball, so it keeps a steady speed, while vertically it rises and falls under gravity exactly as in Year 1. Solve each direction on its own, joined by the time.

MODELLING ASSUMPTION

The ball is modelled as a **particle** with **no air resistance**, moving under a **constant** $g = 9.8 \text{ m s}^{-2}$. Real balls spin and are slowed by the air, so they fall short of these predictions, especially over long distances.

1. Projected horizontally



Example 3.1: evenly spaced across, speeding up downwards. The dots are a quarter of a second apart.

WORKED EXAMPLE 3.1 • THROWN FROM A CLIFF

A ball is thrown horizontally at 12 m s^{-1} from the top of a vertical cliff 20 m high. Find how long it takes to reach the ground, how far from the foot of the cliff it lands, and its speed as it lands.

1. Vertically, it starts with no vertical velocity and falls 20 m:

$$20 = \frac{1}{2} \times 9.8 t^2$$

$$t = 2.02 \text{ s (3 s.f.)}$$

2. Horizontally, it moves at a steady 12 m s^{-1} : $x = 12 \times 2.020 \dots = 24.2 \text{ m}$.

3. Landing: the horizontal velocity is still 12, and the vertical velocity is $9.8 \times 2.020 \dots = 19.79 \dots$, so the speed is $\sqrt{12^2 + 19.79 \dots^2} = 23.2 \text{ m s}^{-1}$ (3 s.f.).

IN CONTEXT

The ball lands 24.2 m from the cliff after 2.02 s, moving at 23.2 m s^{-1} at 58.8° below the horizontal.

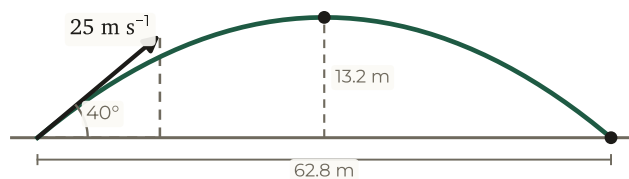
2. Projected at an angle

COMPONENTS OF THE VELOCITY

A particle projected at speed U at an angle α above the horizontal starts with

$U \cos \alpha$ horizontally (constant)

$U \sin \alpha$ vertically (acceleration g down)



Example 3.2: the greatest height comes halfway, and the landing mirrors the launch.

WORKED EXAMPLE 3.2 · TIME, RANGE AND HEIGHT

A ball is kicked from level ground at 25 m s^{-1} at 40° above the horizontal. Find its time of flight, its range and its greatest height.

1. Vertically, with up positive, $u = 25 \sin 40^\circ$ and $a = -9.8$. It lands when the vertical displacement is zero again:

$$0 = 25 \sin 40^\circ t - 4.9t^2$$

$$t = \frac{25 \sin 40^\circ}{4.9} = 3.28 \text{ s (3 s.f.)}$$

2. Horizontally: range = $25 \cos 40^\circ \times 3.279 \dots = 62.8 \text{ m}$.

3. At the top the vertical velocity is zero. From $v^2 = u^2 + 2as$:

$$0 = (25 \sin 40^\circ)^2 - 19.6H$$

$$H = 13.2 \text{ m (3 s.f.)}$$

COMMON MISTAKE

At the top of the flight the **vertical** velocity is zero, but the ball is still moving: its horizontal velocity, $U \cos \alpha$, hasn't changed. The speed at the top is $U \cos \alpha$, not zero.

WORKED EXAMPLE 3.3 · THE VELOCITY AFTER ONE SECOND

For the ball in Example 3.2, find its speed and direction of motion 1 s after it is kicked.

$$v_x = 25 \cos 40^\circ = 19.15 \dots$$

$$v_y = 25 \sin 40^\circ - 9.8 = 6.26 \dots$$

The speed is $\sqrt{v_x^2 + v_y^2} = 20.2 \text{ m s}^{-1}$, at $\tan^{-1} \frac{v_y}{v_x} = 18.1^\circ$ **above** the horizontal: v_y is still positive, so the ball is still rising.

3. The general formulae

Doing Example 3.2 with letters instead of numbers gives formulae for any launch from level ground. You may be asked to derive them, so know the steps, not just the results.

LAUNCH FROM LEVEL GROUND

For speed U at angle α to the horizontal:

$$T = \frac{2U \sin \alpha}{g}$$

$$H = \frac{U^2 \sin^2 \alpha}{2g}$$

$$R = \frac{U^2 \sin 2\alpha}{g}$$

WORKED EXAMPLE 3.4 · DERIVING THE RANGE

Show that the range on level ground is $\frac{U^2 \sin 2\alpha}{g}$, and deduce the angle that gives the greatest range.

1. Vertically, $0 = U \sin \alpha T - \frac{1}{2}gT^2$, so $T = \frac{2U \sin \alpha}{g}$ (the other root, $T = 0$, is the launch).

2. Horizontally,

$$\begin{aligned} R &= U \cos \alpha \times \frac{2U \sin \alpha}{g} \\ &= \frac{U^2 \times 2 \sin \alpha \cos \alpha}{g} \\ &= \frac{U^2 \sin 2\alpha}{g} \end{aligned}$$

using $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ from Chapter 10 of Year 2 Pure.

3. $\sin 2\alpha$ is greatest, 1, when $2\alpha = 90^\circ$. So the greatest range is $\frac{U^2}{g}$, at $\alpha = 45^\circ$.

Because $\sin 2\alpha = \sin(180^\circ - 2\alpha)$, the angles α and $90^\circ - \alpha$ give the **same** range: a high lob and a low drive can land in the same place.

4. The path

The horizontal and vertical positions are both functions of t : a pair of parametric equations, as in Chapter 11 of Year 2 Pure. Eliminating t gives the Cartesian equation of the path.

THE EQUATION OF THE PATH

$$y = x \tan \alpha - \frac{gx^2}{2U^2 \cos^2 \alpha}$$

a parabola, taking the launch point as the origin.

WHY IT WORKS

$x = U \cos \alpha t$ gives $t = \frac{x}{U \cos \alpha}$. Substitute into $y = U \sin \alpha t - \frac{1}{2}gt^2$: the first term becomes $x \tan \alpha$ and the second $\frac{gx^2}{2U^2 \cos^2 \alpha}$.

WORKED EXAMPLE 3.5 · OVER THE WALL?

A ball is thrown from ground level at 14 m s^{-1} at an angle α above the horizontal, where $\tan \alpha = \frac{3}{4}$. Find the equation of its path, and decide whether it clears a wall 2 m high, 10 m away.

1. $\cos \alpha = \frac{4}{5}$, so $2U^2 \cos^2 \alpha = 2 \times 196 \times \frac{16}{25} = 250.88$, and

$$y = \frac{3}{4}x - \frac{9.8x^2}{250.88} = \frac{3}{4}x - \frac{5}{128}x^2$$

2. At $x = 10$: $y = 7.5 - 3.906 \dots = 3.59 \text{ m}$ (3 s.f.).

IN CONTEXT

The ball is 3.59 m high as it reaches the wall, so it clears the 2 m wall.

5. Launched above the ground

When the launch point is above the landing point, the flight doesn't end when the vertical displacement returns to zero: it ends when the displacement equals minus the height. Choose up as positive and keep the sign.

WORKED EXAMPLE 3.6 · THROWN UP FROM A CLIFF

A stone is thrown at 15 m s^{-1} at 30° above the horizontal from the top of a cliff 30 m above the sea. Find how far from the foot of the cliff it enters the water.

1. Vertically, with up positive, it ends 30 m **below** its start:

$$\begin{aligned} -30 &= 7.5t - 4.9t^2 \\ 4.9t^2 - 7.5t - 30 &= 0 \end{aligned}$$

The positive root is $t = 3.355 \dots \text{ s}$.

2. Horizontally: $15 \cos 30^\circ \times 3.355 \dots = 43.6 \text{ m}$ (3 s.f.).

Practice

Twelve questions in three tiers. Try each one properly before opening the solution. The solutions show the full method, not just answers. Take $g = 9.8 \text{ m s}^{-2}$, ignore air resistance, and give answers to 3 significant figures where they aren't exact.

Warm-up**Question 1**

[3 marks]

A stone is thrown horizontally at 8 m s^{-1} from a point 45 m above level ground. Find the time it takes to land and how far it travels horizontally.

SOLUTION

Vertically: $45 = 4.9t^2$, so $t = 3.03 \text{ s}$. Horizontally: $8 \times 3.030 \dots = 24.2 \text{ m}$.

Question 2

[2 marks]

A particle is projected at 30 m s^{-1} at 50° above the horizontal. Find the horizontal and vertical components of its initial velocity.

SOLUTION

Horizontal: $30 \cos 50^\circ = 19.3 \text{ m s}^{-1}$. Vertical: $30 \sin 50^\circ = 23.0 \text{ m s}^{-1}$.

Question 3

[4 marks]

A ball is projected from level ground at 20 m s^{-1} at 30° above the horizontal. Find its time of flight and its range.

SOLUTION

Vertically: $0 = 10t - 4.9t^2$, so $t = \frac{10}{4.9} = 2.04 \text{ s}$.

Horizontally: $20 \cos 30^\circ \times 2.040 \dots = 35.3 \text{ m}$.

Question 4

[3 marks]

A particle is projected at 18 m s^{-1} at 60° above the horizontal. Find its greatest height above the point of projection.

SOLUTION

At the top $v_y = 0$: $0 = (18 \sin 60^\circ)^2 - 19.6H$, so $H = \frac{243}{19.6} = 12.4 \text{ m}$.

Standard**Question 5**

[5 marks]

A ball is projected at 20 m s^{-1} at 35° above the horizontal. Find its speed and direction of motion 1.5 s later.

SOLUTION

$$v_x = 20 \cos 35^\circ = 16.38 \dots$$

$$\begin{aligned} v_y &= 20 \sin 35^\circ - 9.8 \times 1.5 \\ &= -3.228 \dots \end{aligned}$$

The speed is $\sqrt{v_x^2 + v_y^2} = 16.7 \text{ m s}^{-1}$. v_y is negative, so the ball is falling, at $\tan^{-1} \frac{3.228\dots}{16.38\dots} = 11.1^\circ$ below the horizontal.

Question 6

[4 marks]

A particle is projected from level ground at speed U at an angle α above the horizontal. Show that its greatest height is $\frac{U^2 \sin^2 \alpha}{2g}$.

SOLUTION

Vertically the initial velocity is $U \sin \alpha$ and the acceleration is $-g$. At the greatest height the vertical velocity is zero, so by $v^2 = u^2 + 2as$:

$$0 = U^2 \sin^2 \alpha - 2gH$$

$$H = \frac{U^2 \sin^2 \alpha}{2g}$$

Question 7

[5 marks]

A ball is struck from level ground at 25 m s^{-1} and lands 40 m away. Find the two possible angles of projection.

SOLUTION

The range is $\frac{U^2 \sin 2\alpha}{g}$, so

$$\sin 2\alpha = \frac{40 \times 9.8}{625} = 0.6272$$

$2\alpha = 38.84\dots^\circ$ or $180^\circ - 38.84\dots^\circ$, so $\alpha = 19.4^\circ$ or $\alpha = 70.6^\circ$: a low drive or a high lob.

Question 8

[4 marks]

A marble rolls off a horizontal table 1.2 m high and lands on the floor 1.5 m from the edge of the table. Find the speed of the marble as it leaves the table.

SOLUTION

Vertically: $1.2 = 4.9t^2$, so $t = 0.4948\dots$ s. Horizontally the speed is constant: $u = \frac{1.5}{0.4948\dots} = 3.03 \text{ m s}^{-1}$.

Question 9

[4 marks]

A particle is projected from the origin at 10 m s^{-1} at 45° above the horizontal. Find the equation of its path.

SOLUTION

$\tan 45^\circ = 1$ and $\cos^2 45^\circ = \frac{1}{2}$, so

$$\begin{aligned} y &= x - \frac{9.8x^2}{2 \times 100 \times \frac{1}{2}} \\ &= x - 0.098x^2 \end{aligned}$$

Exam style**Question 10**

[9 marks]

A golf ball is hit from level ground at 30 m s^{-1} at an angle α above the horizontal, where $\tan \alpha = \frac{3}{4}$.

- (a) Find the time of flight. [2]
- (b) Find the range. [2]
- (c) Find the greatest height. [2]
- (d) Find for how long the ball is more than 10 m above the ground. [3]

SOLUTION

$\sin \alpha = \frac{3}{5}$ and $\cos \alpha = \frac{4}{5}$, so the components are 18 vertically and 24 horizontally.

(a) $0 = 18t - 4.9t^2$, so $t = \frac{18}{4.9} = 3.67 \text{ s}$.

(b) $24 \times 3.673\dots = 88.2 \text{ m}$.

(c) $0 = 18^2 - 19.6H$, so $H = 16.5 \text{ m}$.

(d) Solve $18t - 4.9t^2 = 10$, that is $4.9t^2 - 18t + 10 = 0$:

$$t = \frac{18 \pm \sqrt{128}}{9.8}$$

$$= 0.682 \dots \text{ or } 2.991 \dots$$

The ball is above 10 m between these times, for 2.31 s.

Question 11

[8 marks]

A stone is thrown at 20 m s^{-1} at 30° above the horizontal from the top of a cliff 25 m above the sea.

- (a) Find the time taken for the stone to reach the sea. [4]
- (b) Find how far from the foot of the cliff it enters the water. [2]
- (c) Find its speed as it enters the water. [2]

SOLUTION

(a) Vertically, with up positive, $u = 10$ and the displacement is -25 :

$$-25 = 10t - 4.9t^2$$

$$4.9t^2 - 10t - 25 = 0$$

The positive root is $t = \frac{10 + \sqrt{590}}{9.8} = 3.50 \text{ s}$.

(b) $20 \cos 30^\circ \times 3.498 \dots = 60.6 \text{ m}$.

(c) $v_x = 17.32 \dots$ and $v_y = 10 - 9.8 \times 3.498 \dots = -24.28 \dots$, so the speed is $\sqrt{v_x^2 + v_y^2} = 29.8 \text{ m s}^{-1}$.

Question 12

[7 marks]

A ball is kicked from ground level at 20 m s^{-1} at 30° above the horizontal, towards a wall 3 m high that is 15 m away.

- (a) Show that the ball passes over the wall. [4]
- (b) Find how far beyond the wall the ball lands. [3]

SOLUTION

(a) The path is $y = x \tan 30^\circ - \frac{9.8x^2}{2 \times 400 \cos^2 30^\circ}$. At $x = 15$:

$$y = 8.660 \dots - 3.675 = 4.99 \text{ m}$$

That is more than 3 m, so the ball clears the wall.

(b) The range is $\frac{400 \sin 60^\circ}{9.8} = 35.3$ m, so it lands $35.3 - 15 = 20.3$ m beyond the wall.