

YEAR 2 MECHANICS · CHAPTER 5

Further Kinematics

Specification: Edexcel Paper 3 §7.3, 7.4 — Kinematics in two dimensions · AQA section Q3, Q4

KEY FACTS — THIS CHAPTER

- In two dimensions, position $\mathbf{r}$, velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are **vectors**, written with $\mathbf{i}$ and $\mathbf{j}$.
- **Speed** is the magnitude of the velocity: for $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ it is $\sqrt{x^2 + y^2}$. The **direction of motion** is the direction of $\mathbf{v}$.
- **Constant acceleration:** $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$, the Year 1 formulae with vectors in place of numbers.
- **Variable acceleration:** differentiate $\mathbf{r}$ to get $\mathbf{v}$, and $\mathbf{v}$ to get $\mathbf{a}$, one component at a time. Integrate to go back, with a **vector** constant of integration.
- A particle moves **parallel to $\mathbf{i}$** when the $\mathbf{j}$ -component of its velocity is zero, and parallel to $\mathbf{j}$ when the $\mathbf{i}$ -component is zero.
- A projectile is constant acceleration with $\mathbf{a} = -g\mathbf{j}$.

Year 1 kinematics kept everything on one line: a single number told you how far along the line a particle was. A ship at sea, a drone or a ball in flight needs two numbers, so position becomes a vector. The good news is that nothing else is new. The suvat formulae and the calculus of Year 1 work exactly as before, applied to each component separately.

MODELLING ASSUMPTION

$\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors. Unless a question says otherwise, $\mathbf{i}$ points due **east** and $\mathbf{j}$ due **north**; for motion under gravity, $\mathbf{i}$ is horizontal and $\mathbf{j}$ vertically **up**. Time t is measured in seconds from the start of the motion.

1. Constant acceleration in two dimensions

CONSTANT ACCELERATION AS VECTORS

A particle with initial velocity $\mathbf{u}$ and constant acceleration $\mathbf{a}$, starting at the point with position vector $\mathbf{r}_0$, has at time t

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

WHY IT WORKS

Each component moves with constant acceleration on its own line, so each obeys the Year 1 formulae. Writing the two components side by side as one vector equation gives these results; $\mathbf{r}_0$ is there because the particle need not start at the origin.

WORKED EXAMPLE 5.1 • VELOCITY, SPEED AND BEARING

A particle has initial velocity $(2\mathbf{i} - 3\mathbf{j}) \text{ m s}^{-1}$ and constant acceleration $(\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-2}$. Find its speed after 4 s and the bearing on which it is then moving. Find also when it is moving parallel to $\mathbf{i}$.

1. From $\mathbf{v} = \mathbf{u} + \mathbf{a}t$:

$$\mathbf{v} = (2 + t)\mathbf{i} + (2t - 3)\mathbf{j}$$

2. At $t = 4$, $\mathbf{v} = 6\mathbf{i} + 5\mathbf{j}$, so the speed is $\sqrt{6^2 + 5^2} = \sqrt{61} = 7.81 \text{ m s}^{-1}$ (3 s.f.).
3. The velocity points $\tan^{-1} \frac{5}{6} = 39.8^\circ$ north of east, so the bearing is $90^\circ - 39.8^\circ = 050.2^\circ$.
4. Parallel to $\mathbf{i}$, the $\mathbf{j}$ -component is zero: $2t - 3 = 0$, so $t = 1.5 \text{ s}$.

COMMON MISTAKE

Speed is $|\mathbf{v}|$, found by Pythagoras. It is **not** the sum of the components, and it is never negative. A velocity of $-4\mathbf{j}$ is a speed of 4 m s^{-1} due south.

WORKED EXAMPLE 5.2 · DUE NORTH OF THE ORIGIN

A particle starts at the point with position vector $(4\mathbf{i} + 2\mathbf{j})$ m, with velocity $(3\mathbf{i} - \mathbf{j})$ m s⁻¹ and constant acceleration $(-\mathbf{i} + 2\mathbf{j})$ m s⁻². Find its distance from the origin O after 2 s, and the time at which it is due north of O .

1. From $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$:

$$\mathbf{r} = (4 + 3t - \frac{1}{2}t^2)\mathbf{i} + (2 - t + t^2)\mathbf{j}$$

2. At $t = 2$, $\mathbf{r} = 8\mathbf{i} + 4\mathbf{j}$, a distance $\sqrt{8^2 + 4^2} = \sqrt{80} = 8.94$ m (3 s.f.) from O .

3. Due north of O , the $\mathbf{i}$ -component is zero:

$$4 + 3t - \frac{1}{2}t^2 = 0$$

$$t^2 - 6t - 8 = 0$$

so $t = 3 + \sqrt{17} = 7.12$ s (3 s.f.), taking the positive root.

IN CONTEXT

After 7.12 s the particle is due north of O . Its $\mathbf{j}$ -component is then $25 + 5\sqrt{17} = 45.6$, so it is 45.6 m north of O .

Constant **velocity** is the special case $\mathbf{a} = \mathbf{0}$: $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$. Two ships on steady courses each follow such a line, and they meet only if they reach the **same** point at the **same** time.

WORKED EXAMPLE 5.3 · DO THE SHIPS MEET?

At noon ship A is at $(\mathbf{i} + 2\mathbf{j})$ km and ship B is at $(7\mathbf{i} - 4\mathbf{j})$ km. A sails at a constant $(3\mathbf{i} + \mathbf{j})$ km h⁻¹ and B at a constant $(\mathbf{i} + 3\mathbf{j})$ km h⁻¹. Show that, if nothing changes, the ships collide, and find where.

1. t hours after noon:

$$\mathbf{r}_A = (1 + 3t)\mathbf{i} + (2 + t)\mathbf{j}$$

$$\mathbf{r}_B = (7 + t)\mathbf{i} + (3t - 4)\mathbf{j}$$

2. The $\mathbf{i}$ -components are equal when $1 + 3t = 7 + t$, that is at $t = 3$.
3. At $t = 3$ the $\mathbf{j}$ -components are $2 + 3 = 5$ and $9 - 4 = 5$: equal as well. So $\mathbf{r}_A = \mathbf{r}_B = 10\mathbf{i} + 5\mathbf{j}$ at the same moment.

IN CONTEXT

The ships collide at 3 pm, at the point $(10\mathbf{i} + 5\mathbf{j})$ km.

COMMON MISTAKE

Crossing paths is not a collision. Find the time that makes **one** pair of components equal, then check the **other** pair at that same time. If they differ, the ships pass through the same point at different times, and never meet.

2. Variable acceleration: calculus with vectors

When the acceleration changes, the suvat formulae no longer apply. As in Year 1, velocity is the rate of change of position and acceleration the rate of change of velocity, and a vector is differentiated one component at a time.

DIFFERENTIATING AND INTEGRATING VECTORS

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

and so $\mathbf{v} = \int \mathbf{a} dt$ and $\mathbf{r} = \int \mathbf{v} dt$, each with a vector constant of integration.

WORKED EXAMPLE 5.4 · DIFFERENTIATING A POSITION VECTOR

A particle moves so that at time t s its position vector is $\mathbf{r} = (t^3 - 3t)\mathbf{i} + (2t^2 - 8t)\mathbf{j}$ m. Find when it is moving parallel to $\mathbf{j}$, and its velocity then. Find also the magnitude of its acceleration at that time.

1. Differentiate each component:

$$\mathbf{v} = (3t^2 - 3)\mathbf{i} + (4t - 8)\mathbf{j}$$

$$\mathbf{a} = 6t\mathbf{i} + 4\mathbf{j}$$

2. Parallel to $\mathbf{j}$, the $\mathbf{i}$ -component is zero: $3t^2 - 3 = 0$, so $t = 1$ (time is positive). Then $\mathbf{v} = -4\mathbf{j}$.
3. At $t = 1$, $\mathbf{a} = 6\mathbf{i} + 4\mathbf{j}$, with magnitude $\sqrt{52} = 7.21 \text{ m s}^{-2}$ (3 s.f.).

IN CONTEXT

At $t = 1$ the particle moves due south at 4 m s^{-1} . A second later, $\mathbf{v} = 9\mathbf{i}$: it is moving due east.

WORKED EXAMPLE 5.5 • INTEGRATING AN ACCELERATION

A particle starts at the origin with velocity $(2\mathbf{i} + 5\mathbf{j}) \text{ m s}^{-1}$. Its acceleration at time t s is $(6t\mathbf{i} - 4\mathbf{j}) \text{ m s}^{-2}$. Find its position vector when it is moving due east.

1. Integrate, with a constant vector $\mathbf{c}$: $\mathbf{v} = 3t^2\mathbf{i} - 4t\mathbf{j} + \mathbf{c}$. At $t = 0$, $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$, so $\mathbf{c} = 2\mathbf{i} + 5\mathbf{j}$:

$$\mathbf{v} = (3t^2 + 2)\mathbf{i} + (5 - 4t)\mathbf{j}$$

2. Integrate again. It starts at the origin, so this constant is $\mathbf{0}$:

$$\mathbf{r} = (t^3 + 2t)\mathbf{i} + (5t - 2t^2)\mathbf{j}$$

3. Due east means the $\mathbf{j}$ -component of $\mathbf{v}$ is zero while the $\mathbf{i}$ -component is positive: $5 - 4t = 0$, so $t = 1.25$, and $3t^2 + 2 > 0$ always.
4. At $t = 1.25$:

$$\begin{aligned}\mathbf{r} &= \frac{285}{64}\mathbf{i} + \frac{25}{8}\mathbf{j} \\ &= 4.45\mathbf{i} + 3.13\mathbf{j} \text{ (3 s.f.)}\end{aligned}$$

COMMON MISTAKE

The constant of integration is a **vector**. Find both of its components from the starting condition. Dropping it, or adding a single number, silently puts the particle in the wrong place.

3. Projectiles as vectors

A projectile from Chapter 3 is a particle with constant acceleration $-g\mathbf{j}$, so the vector formulae describe the whole flight in one line each. The two methods are the same calculation: the vector form just keeps the horizontal and vertical parts together.

WORKED EXAMPLE 5.6 • A PROJECTILE IN VECTOR FORM

A particle is projected from the origin with velocity $(6\mathbf{i} + 8\mathbf{j}) \text{ m s}^{-1}$ and moves freely under gravity. Find its position vector at the moment it is moving at 45° below the horizontal.

1. With $\mathbf{a} = -9.8\mathbf{j}$:

$$\mathbf{v} = 6\mathbf{i} + (8 - 9.8t)\mathbf{j}$$

$$\mathbf{r} = 6t\mathbf{i} + (8t - 4.9t^2)\mathbf{j}$$

2. At 45° below the horizontal the components have equal size, with the $\mathbf{j}$ -component negative:

$$8 - 9.8t = -6, \text{ so } t = \frac{10}{9.8}.$$

3. Then $\mathbf{r} = \frac{60}{9.8}\mathbf{i} + \frac{10}{9.8}\mathbf{j} = 8.57\mathbf{i} + 1.43\mathbf{j} \text{ m (3 s.f.)}$.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution. The solutions show the full method, not just answers. $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north, unless a question says otherwise. Take $g = 9.8 \text{ m s}^{-2}$ and give answers to 3 significant figures where they aren't exact.

Warm-up**Question 1**

[3 marks]

A particle has initial velocity $(3\mathbf{i} - 2\mathbf{j}) \text{ m s}^{-1}$ and constant acceleration $(2\mathbf{i} + \mathbf{j}) \text{ m s}^{-2}$. Find its velocity and its speed after 5 s.

SOLUTION

$$\mathbf{v} = (3\mathbf{i} - 2\mathbf{j}) + 5(2\mathbf{i} + \mathbf{j}) = 13\mathbf{i} + 3\mathbf{j}, \text{ so the speed is } \sqrt{13^2 + 3^2} = \sqrt{178} = 13.3 \text{ m s}^{-1}.$$

Question 2

[3 marks]

A particle starts at the origin with velocity $(4\mathbf{i} + \mathbf{j}) \text{ m s}^{-1}$ and has constant acceleration $-2\mathbf{j} \text{ m s}^{-2}$. Find its position vector after 3 s and its distance from the origin then.

SOLUTION

$\mathbf{r} = 3(4\mathbf{i} + \mathbf{j}) + \frac{1}{2} \times 9 \times (-2\mathbf{j}) = 12\mathbf{i} - 6\mathbf{j}$, a distance $\sqrt{180} = 13.4$ m from the origin.

Question 3

[4 marks]

A particle's position vector at time t s is $\mathbf{r} = (2t^2 - 3t)\mathbf{i} + (t^3 + 1)\mathbf{j}$ m. Find its speed and its acceleration when $t = 2$.

SOLUTION

$$\mathbf{v} = (4t - 3)\mathbf{i} + 3t^2\mathbf{j}$$

$$\mathbf{a} = 4\mathbf{i} + 6t\mathbf{j}$$

At $t = 2$: $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$, so the speed is $\sqrt{25 + 144} = 13 \text{ m s}^{-1}$, and $\mathbf{a} = (4\mathbf{i} + 12\mathbf{j}) \text{ m s}^{-2}$.

Question 4

[2 marks]

A boat moves with velocity $(-5\mathbf{i} + 5\mathbf{j}) \text{ km h}^{-1}$. Find its speed and the bearing on which it is moving.

SOLUTION

The speed is $\sqrt{25 + 25} = 5\sqrt{2} = 7.07 \text{ km h}^{-1}$. It moves equally west and north, that is north-west, on a bearing of 315° .

Standard**Question 5**

[5 marks]

A particle starts from rest at the point with position vector $(2\mathbf{i} - \mathbf{j})$ m and moves with constant acceleration $(3\mathbf{i} + 4\mathbf{j}) \text{ m s}^{-2}$. Find its speed after 4 s, its position vector then, and how far it is from its starting point.

SOLUTION

$\mathbf{v} = 4(3\mathbf{i} + 4\mathbf{j}) = 12\mathbf{i} + 16\mathbf{j}$, a speed of $\sqrt{144 + 256} = 20 \text{ m s}^{-1}$.

$$\begin{aligned}\mathbf{r} &= (2\mathbf{i} - \mathbf{j}) + \frac{1}{2} \times 16(3\mathbf{i} + 4\mathbf{j}) \\ &= 26\mathbf{i} + 31\mathbf{j}\end{aligned}$$

It has moved $24\mathbf{i} + 32\mathbf{j}$, a distance of $\sqrt{576 + 1024} = 40$ m.

Question 6

[5 marks]

At noon ship A is at $(2\mathbf{i} + 3\mathbf{j})$ km and ship B at $(10\mathbf{i} - \mathbf{j})$ km. A has constant velocity $(4\mathbf{i} - \mathbf{j})$ km h⁻¹ and B constant velocity $(2\mathbf{i} + \mathbf{j})$ km h⁻¹. Find when B is due north of A , and how far apart the ships are then.

SOLUTION

t hours after noon,

$$\mathbf{r}_A = (2 + 4t)\mathbf{i} + (3 - t)\mathbf{j}$$

$$\mathbf{r}_B = (10 + 2t)\mathbf{i} + (t - 1)\mathbf{j}$$

so the position of B relative to A is $\mathbf{r}_B - \mathbf{r}_A = (8 - 2t)\mathbf{i} + (2t - 4)\mathbf{j}$.

Due north means no $\mathbf{i}$ -component: $8 - 2t = 0$, so $t = 4$, at 4 pm. Then $\mathbf{r}_B - \mathbf{r}_A = 4\mathbf{j}$: the ships are 4 km apart.

Question 7

[5 marks]

A particle has velocity $\mathbf{v} = (6t^2 - 4)\mathbf{i} + (3 - 2t)\mathbf{j}$ m s⁻¹ at time t s. At $t = 0$ it is at the point with position vector $\mathbf{i}$ m. Find when it is moving parallel to $\mathbf{i}$, and its position vector when $t = 3$.

SOLUTION

Parallel to $\mathbf{i}$: $3 - 2t = 0$, so $t = 1.5$ s.

Integrating, $\mathbf{r} = (2t^3 - 4t)\mathbf{i} + (3t - t^2)\mathbf{j} + \mathbf{c}$, and $\mathbf{r} = \mathbf{i}$ at $t = 0$ gives $\mathbf{c} = \mathbf{i}$:

$$\mathbf{r} = (2t^3 - 4t + 1)\mathbf{i} + (3t - t^2)\mathbf{j}$$

At $t = 3$: $\mathbf{r} = 43\mathbf{i}$ m.

Question 8

[5 marks]

A particle has acceleration $(2\mathbf{i} - 3t\mathbf{j})$ m s⁻² at time t s, and initial velocity $(\mathbf{i} + 4\mathbf{j})$ m s⁻¹. Find the time at which it is moving due east.

SOLUTION

$\mathbf{v} = (2t + 1)\mathbf{i} + (4 - \frac{3}{2}t^2)\mathbf{j}$, using the initial velocity for the constant.

Due east: the $\mathbf{j}$ -component is zero and the $\mathbf{i}$ -component positive. $4 - \frac{3}{2}t^2 = 0$ gives $t^2 = \frac{8}{3}$, so $t = 1.63$ s, when $2t + 1 > 0$.

Question 9

[5 marks]

A particle of mass 2 kg starts from rest at the origin. At time t s it is acted on by the single force $\mathbf{F} = (6\mathbf{i} - 4t\mathbf{j})$ N. Find its speed after 3 s and the bearing on which it is then moving.

SOLUTION

By $\mathbf{F} = m\mathbf{a}$, $\mathbf{a} = 3\mathbf{i} - 2t\mathbf{j}$. Starting from rest, $\mathbf{v} = 3t\mathbf{i} - t^2\mathbf{j}$.

At $t = 3$: $\mathbf{v} = 9\mathbf{i} - 9\mathbf{j}$, a speed of $9\sqrt{2} = 12.7 \text{ m s}^{-1}$. It moves equally east and south, on a bearing of 135° .

Exam style**Question 10**

[9 marks]

In this question $\mathbf{i}$ is horizontal and $\mathbf{j}$ vertically upwards. A ball is projected from a point O on level ground with velocity $(14\mathbf{i} + 19.6\mathbf{j}) \text{ m s}^{-1}$.

- (a) Find the position vector of the ball t seconds after projection. [2]
- (b) Find the time of flight and the range. [3]
- (c) Find the speed of the ball 3 s after projection, and the angle its direction of motion then makes with the horizontal. [4]

SOLUTION

(a) With $\mathbf{a} = -9.8\mathbf{j}$: $\mathbf{r} = 14t\mathbf{i} + (19.6t - 4.9t^2)\mathbf{j}$.

(b) It lands when $19.6t - 4.9t^2 = 0$, so $t = 4$ s. The range is $14 \times 4 = 56$ m.

(c) $\mathbf{v} = 14\mathbf{i} + (19.6 - 9.8t)\mathbf{j}$, so at $t = 3$, $\mathbf{v} = 14\mathbf{i} - 9.8\mathbf{j}$. The speed is $\sqrt{14^2 + 9.8^2} = 17.1 \text{ m s}^{-1}$, moving at $\tan^{-1} \frac{9.8}{14} = 35.0^\circ$ below the horizontal.

Question 11

[9 marks]

A particle P has velocity $\mathbf{v} = (3t^2 - 12)\mathbf{i} + (2t - 5)\mathbf{j} \text{ m s}^{-1}$ at time t s. When $t = 0$, P is at the point with position vector $(4\mathbf{i} + 2\mathbf{j}) \text{ m}$.

- (a) Find the acceleration of P when $t = 1$. [2]
- (b) Find the time at which P is moving parallel to $\mathbf{j}$, and its speed then. [3]
- (c) Find the distance of P from the origin at that time. [4]

SOLUTION

(a) $\mathbf{a} = 6t\mathbf{i} + 2\mathbf{j}$, so at $t = 1$, $\mathbf{a} = (6\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-2}$.

(b) Parallel to $\mathbf{j}$: $3t^2 - 12 = 0$, so $t = 2$. Then $\mathbf{v} = -\mathbf{j}$, a speed of 1 m s^{-1} .

(c) Integrating, with the starting point for the constant:

$$\mathbf{r} = (t^3 - 12t + 4)\mathbf{i} + (t^2 - 5t + 2)\mathbf{j}$$

At $t = 2$: $\mathbf{r} = -12\mathbf{i} - 4\mathbf{j}$, a distance of $\sqrt{160} = 12.6 \text{ m}$ from the origin.

Question 12

[8 marks]

Particle A starts at the origin and moves with constant velocity $(3\mathbf{i} + 4\mathbf{j}) \text{ m s}^{-1}$. At the same moment particle B starts at the point $(10\mathbf{i} + 2\mathbf{j}) \text{ m}$ and moves with constant velocity $(-\mathbf{i} + 3\mathbf{j}) \text{ m s}^{-1}$.

(a) Show that the square of the distance between them at time $t \text{ s}$ is $17t^2 - 84t + 104$. [4]

(b) Find the time at which they are closest, and the least distance between them. [4]

SOLUTION

(a) $\mathbf{r}_A = 3t\mathbf{i} + 4t\mathbf{j}$ and $\mathbf{r}_B = (10 - t)\mathbf{i} + (2 + 3t)\mathbf{j}$, so

$$\mathbf{r}_B - \mathbf{r}_A = (10 - 4t)\mathbf{i} + (2 - t)\mathbf{j}$$

The square of the distance is

$$\begin{aligned} d^2 &= (10 - 4t)^2 + (2 - t)^2 \\ &= 17t^2 - 84t + 104 \end{aligned}$$

(b) d^2 is least when its derivative is zero: $34t - 84 = 0$, so $t = \frac{42}{17} = 2.47 \text{ s}$. Then $d^2 = \frac{4}{17}$, so the least distance is $\sqrt{\frac{4}{17}} = 0.485 \text{ m}$.