

YEAR 2 MECHANICS · CHAPTER 4

Applications of Forces

Specification: Edexcel Paper 3 §8.4, 8.6, 9.1 — Statics, friction and moments · AQA section R4, R6, S1

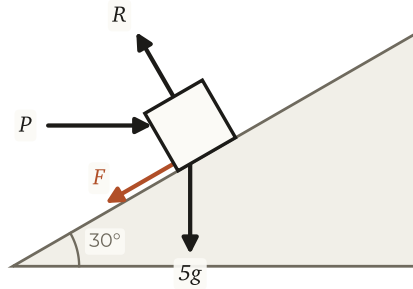
KEY FACTS — THIS CHAPTER

- Every problem in this chapter uses the same three tools: **resolve** in two perpendicular directions, the friction model $F \leq \mu R$, and, for a rigid body, **moments**.
- Friction opposes the motion that would happen without it. Decide which way the body is about to move **first**; the direction of friction follows.
- A **rigid body** is in equilibrium when the forces balance in two directions **and** the moments balance about any one point. Take moments about the point where most of the unknown forces act.
- **Ladders:** a smooth wall pushes at right angles to itself only; rough ground gives a normal reaction and friction towards the wall. On the point of slipping, $F = \mu R$.
- **Connected particles:** write an equation of motion for each particle, joined by the tension and the shared acceleration. First check that the system moves at all.
- When a string goes slack its tension is zero, and each particle moves on its own under the forces that remain.

Chapters 1 and 2 built the tools one at a time: moments for rigid bodies, resolving and friction for particles. Real problems need them together. A ladder leans on a wall and relies on friction at its foot; a crate is dragged up a rough ramp by a falling weight. This chapter puts the tools to work, and the skill it builds is choosing which one to use at each step.

1. Particles in limiting equilibrium

A force that pushes at an angle to a surface does two jobs: part of it drives the particle along, and part presses it into the surface or lifts it away. That second part changes R , and with it the greatest friction μR . So find R first, with every force included.



Example 4.1(a): about to move up the plane, so friction acts down it.

WORKED EXAMPLE 4.1 · PUSHED HORIZONTALLY

A particle of mass 5 kg rests on a rough plane inclined at 30° to the horizontal, with coefficient of friction 0.2. It is held by a horizontal force P in the vertical plane through a line of greatest slope. Find P when the particle is on the point of moving (a) up the plane and (b) down the plane.

1. Perpendicular to the plane, P presses the particle into it as well:

$$R = 5g \cos 30^\circ + P \sin 30^\circ$$

so the greatest friction is $0.2R = 8.487 \dots + 0.1P$. The weight's component down the plane is $5g \sin 30^\circ = 24.5$ N.

2. (a) About to move up: friction acts **down** the plane. Along the plane,

$$\begin{aligned} P \cos 30^\circ &= 24.5 + 0.2R \\ &= 32.98 \dots + 0.1P \end{aligned}$$

so $0.766 \dots P = 32.98 \dots$ and $P = 43.1$ N (3 s.f.).

3. (b) About to slide down: friction acts **up** the plane. Along the plane,

$$\begin{aligned} P \cos 30^\circ + 0.2R &= 24.5 \\ 0.966 \dots P + 8.487 \dots &= 24.5 \end{aligned}$$

so $0.966 \dots P = 16.01 \dots$ and $P = 16.6$ N (3 s.f.).

IN CONTEXT

Any horizontal force from 16.6 N to 43.1 N holds the particle still. Pushing up the plane costs more than it would with a force parallel to the plane, because part of the push, $P \sin 30^\circ$, presses the particle in and raises the friction.

COMMON MISTAKE

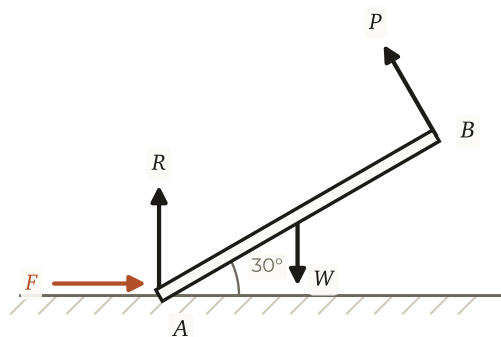
Friction has no fixed direction. It opposes the motion that is **about** to happen, so it flips between parts (a) and (b), and its size μR changes too, because R depends on P . Draw a separate force diagram for each case.

2. Rigid bodies with friction

A rigid body resting on rough ground can slide as well as turn. The conditions from Chapter 1 still hold, and friction adds one more unknown and one more fact.

A RIGID BODY IN LIMITING EQUILIBRIUM

Resolve horizontally, resolve vertically, and take moments about one point: three equations. Where the body is on the point of slipping, add $F = \mu R$ at that contact.



Example 4.2: P tips the rod back towards A , so friction pushes the foot the other way.

WORKED EXAMPLE 4.2 · THE LEAST COEFFICIENT OF FRICTION

A uniform rod AB of length $2a$ and weight W rests with A on rough horizontal ground. It is held at 30° to the ground by a force P at B , acting at right angles to the rod in the vertical plane containing it. Find P in terms of W , and the least possible value of the coefficient of friction.

1. Take moments about A , where both unknown ground forces act. P is at right angles to the rod, $2a$ from A ; the weight acts at the midpoint, a horizontal distance $a \cos 30^\circ$ from A :

$$P \times 2a = W \times a \cos 30^\circ$$

$$P = \frac{\sqrt{3}}{4}W$$

2. P makes 30° with the vertical. Resolve horizontally and vertically:

$$F = P \sin 30^\circ = \frac{\sqrt{3}}{8}W$$

$$R = W - P \cos 30^\circ = \frac{5}{8}W$$

3. The rod rests only if $F \leq \mu R$, so $\mu \geq \frac{\sqrt{3}/8}{5/8} = \frac{\sqrt{3}}{5}$.

IN CONTEXT

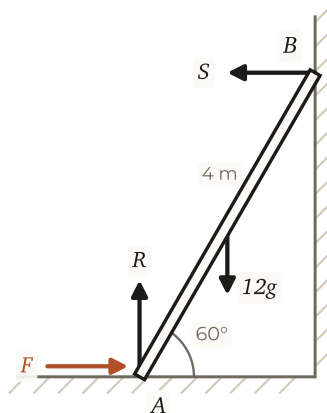
The coefficient of friction must be at least $\frac{\sqrt{3}}{5} = 0.346$ (3 s.f.). The length $2a$ and the weight W both cancel, so the answer holds for any such rod.

3. Ladders

The standard ladder problem is the model at its clearest. The ladder is a uniform rod. The **wall is smooth**, so it pushes at right angles to itself, horizontally and away from the wall. The **ground is rough**, so it gives an upward reaction R and a friction force F . If the ladder slipped, its foot would slide out from the wall, so friction acts **towards** the wall.

MODELLING ASSUMPTION

The ladder is a **uniform rod**: rigid, straight, with its weight at the midpoint. The wall is **smooth**. A climber is a **particle** at one point of the ladder. A real wall has some friction, which helps: the ladder is a little safer than the model predicts.



Example 4.3: four forces on the ladder. Two of them act at the foot.

WORKED EXAMPLE 4.3 · ON THE POINT OF SLIPPING

A uniform ladder AB , of length 4 m and mass 12 kg, rests with its foot A on rough horizontal ground and its top B against a smooth vertical wall. It makes 60° with the ground and is on the point of slipping. Find the coefficient of friction between the ladder and the ground.

1. Vertically: $R = 12g$.
2. Moments about A . The wall's force S acts at B , a height $4 \sin 60^\circ$ above A ; the weight acts a horizontal distance $2 \cos 60^\circ$ from A :

$$S \times 4 \sin 60^\circ = 12g \times 2 \cos 60^\circ$$

$$S = 2\sqrt{3}g = 33.9 \text{ N}$$

3. Horizontally $F = S$, and on the point of slipping $F = \mu R$:

$$\mu = \frac{2\sqrt{3}g}{12g} = \frac{\sqrt{3}}{6} = 0.289 \text{ (3 s.f.)}$$

The length and the mass cancelled. With letters, the same steps show that a uniform ladder on its own is on the point of slipping when $\tan \theta = \frac{1}{2\mu}$, where θ is its angle to the ground. Stand it any steeper and it holds; the smaller μ , the steeper it must stand.

COMMON MISTAKE

Take moments about the **foot**, not the top. Two unknowns, R and F , act at the foot, so both drop out of the moments equation there. About the top only S drops out, and you are left with two unknowns in one equation.

WORKED EXAMPLE 4.4 · HOW FAR CAN HE CLIMB?

A uniform ladder of length 5 m and mass 20 kg rests on rough horizontal ground against a smooth vertical wall, at an angle α to the ground, where $\tan \alpha = \frac{4}{3}$. The coefficient of friction is 0.4. A man of mass 80 kg climbs the ladder. How far up the ladder can he climb before it slips?

1. Vertically: $R = 20g + 80g = 100g$, so friction can be at most $0.4 \times 100g = 40g$. On the point of slipping $F = 40g$, and horizontally the wall's force is $S = F = 40g$.
2. $\sin \alpha = \frac{4}{5}$ and $\cos \alpha = \frac{3}{5}$. With the man x m up the ladder, measured from the foot, the distances from the foot are: - the top: $5 \sin \alpha = 4$ m up; - the ladder's centre: $2.5 \cos \alpha = 1.5$ m across; - the man: $x \cos \alpha = 0.6x$ m across.
3. Take moments about the foot and divide through by g :

$$40 \times 4 = 20 \times 1.5 + 80 \times 0.6x$$

$$160 = 30 + 48x$$

$$x = 2.71 \text{ m (3 s.f.)}$$

IN CONTEXT

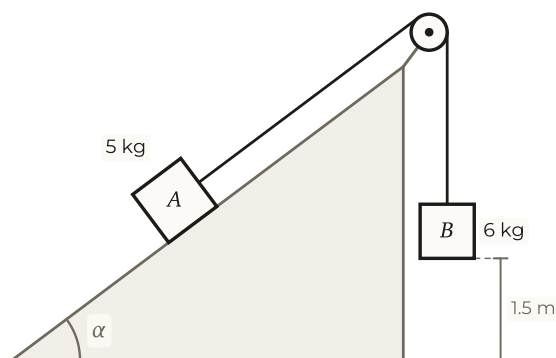
He can climb 2.71 m, just over halfway. As he climbs, R stays at $100g$, but the wall must push harder to balance his growing moment, so the foot needs more friction. At 2.71 m it needs all $40g$ the ground can give.

4. Connected particles

Year 1 joined particles by strings over smooth pulleys, on smooth surfaces. Add slopes and friction and the method is the same: an equation of motion for each particle, joined by the tension T and the shared acceleration a . There is one new first step.

DOES IT MOVE?

Before finding an acceleration, compare the force trying to move the system with the greatest friction that can resist it. If the friction can match it, the system stays at rest, and the friction is only as large as it needs to be.



Examples 4.5 and 4.6: B falls, pulling A up the rough slope.

WORKED EXAMPLE 4.5 · UP A ROUGH SLOPE

Particle A, of mass 5 kg, rests on a rough plane inclined at α to the horizontal, where $\tan \alpha = \frac{3}{4}$. The coefficient of friction is $\frac{1}{4}$. A light inextensible string, parallel to a line of greatest slope, joins A over a smooth pulley at the top of the plane to particle B, of mass 6 kg, which hangs freely. The system is released from rest. Find the acceleration and the tension in the string.

1. For A: $R = 5g \cos \alpha = 4g$, so friction is at most $\frac{1}{4} \times 4g = g$. The weight down the plane is $5g \sin \alpha = 3g$.
2. Does it move? B pulls with $6g$, more than $3g + g = 4g$, so B falls and A moves up the plane, with limiting friction g acting down it.
3. An equation of motion for each particle:

$$6g - T = 6a \quad (B)$$

$$T - 3g - g = 5a \quad (A)$$

4. Adding eliminates T : $2g = 11a$, so $a = \frac{2g}{11} = 1.78 \text{ m s}^{-2}$ (3 s.f.). Then $T = 6(g - a) = \frac{54g}{11} = 48.1 \text{ N}$ (3 s.f.).

WORKED EXAMPLE 4.6 · WHEN THE STRING GOES SLACK

In Example 4.5, B starts 1.5 m above the ground and does not rebound. Find how much further A moves up the plane, and what happens to it next.

1. B falls 1.5 m from rest at $\frac{2g}{11}$: $v^2 = 2 \times \frac{2g}{11} \times 1.5 = \frac{6g}{11}$, so $v = 2.31 \text{ m s}^{-1}$ (3 s.f.).
2. Then the string goes slack. With $T = 0$, the weight component $3g$ and the friction g both act down the plane on A , so it decelerates at $\frac{4g}{5} = 7.84 \text{ m s}^{-2}$.
3. From $v^2 = u^2 + 2as$ with final speed zero:

$$0 = \frac{6g}{11} - 2 \times \frac{4g}{5} s$$

$$s = \frac{15}{44} = 0.341 \text{ m (3 s.f.)}$$

4. At rest, $\tan \alpha = 0.75$ is more than $\mu = 0.25$, so friction cannot hold A on the plane.

IN CONTEXT

A moves a further 0.341 m up the plane, stops for an instant, then slides back down.

COMMON MISTAKE

Once the string is slack, don't carry on using the acceleration you found for the connected system. The forces on A have changed: the tension has gone, and if A turns round, friction changes direction too.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution. The solutions show the full method, not just answers. Take $g = 9.8 \text{ m s}^{-2}$ and give answers to 3 significant figures where they aren't exact. Ladders and rods are uniform, strings are light and inextensible, and pulleys are smooth, unless a question says otherwise.

Warm-up**Question 1**

[4 marks]

A box of mass 3 kg rests on rough horizontal ground, with coefficient of friction 0.4. It is pushed by a force P acting at 30° **below** the horizontal. Find P when the box is on the point of moving.

SOLUTION

Vertically, the push adds to the weight: $R = 3g + P \sin 30^\circ = 29.4 + 0.5P$.

Horizontally, on the point of moving:

$$\begin{aligned} P \cos 30^\circ &= 0.4(29.4 + 0.5P) \\ 0.666 \dots P &= 11.76 \end{aligned}$$

so $P = 17.7 \text{ N}$.

Question 2

[4 marks]

A ladder of length 6 m and mass 15 kg rests on rough horizontal ground against a smooth vertical wall, at 70° to the ground. Find the force exerted by the wall, and the friction and normal reaction at the foot.

SOLUTION

Moments about the foot:

$$\begin{aligned} S \times 6 \sin 70^\circ &= 15g \times 3 \cos 70^\circ \\ S &= 26.8 \text{ N} \end{aligned}$$

Horizontally $F = S = 26.8 \text{ N}$, and vertically $R = 15g = 147 \text{ N}$.

Question 3

[4 marks]

Particle A , of mass 2 kg, lies on a smooth plane inclined at 30° . A string parallel to the plane joins it, over a pulley at the top, to particle B , of mass 3 kg, which hangs freely. The system is released from rest. Find the acceleration and the tension.

SOLUTION

$$\begin{aligned} 3g - T &= 3a \quad (B) \\ T - 2g \sin 30^\circ &= 2a \quad (A) \end{aligned}$$

Adding: $2g = 5a$, so $a = 3.92 \text{ m s}^{-2}$, and $T = 3(g - a) = 17.6 \text{ N}$.

Question 4

[3 marks]

A ladder rests on rough horizontal ground, with coefficient of friction 0.4, against a smooth vertical wall. Find the smallest angle to the ground at which it can rest.

SOLUTION

A ladder on its own is on the point of slipping when $\tan \theta = \frac{1}{2\mu} = \frac{1}{0.8} = 1.25$, so $\theta = 51.3^\circ$. Any smaller angle and it slips.

Standard**Question 5**

[5 marks]

A particle of mass 2 kg rests on a rough plane inclined at 30° , with coefficient of friction 0.4. A string pulls it with tension T at 20° **above** a line of greatest slope, up the plane. Find T when the particle is on the point of moving up the plane.

SOLUTION

The string lifts the particle a little: $R = 2g \cos 30^\circ - T \sin 20^\circ$.

About to move up, friction $0.4R$ acts down the plane:

$$\begin{aligned} T \cos 20^\circ &= 2g \sin 30^\circ + 0.4R \\ 1.076 \dots T &= 9.8 + 6.789 \dots \end{aligned}$$

so $T = 15.4$ N.

Question 6

[5 marks]

A ladder of length 4 m and mass 10 kg rests at 60° to rough horizontal ground, with coefficient of friction 0.35, against a smooth vertical wall. A woman of mass 70 kg climbs the ladder. How far up can she go before it slips?

SOLUTION

$R = 80g$, so on the point of slipping $F = 0.35 \times 80g = 28g$, and $S = 28g$. Take moments about the foot, with the woman x m up, and divide through by g . The ladder's weight acts $2 \cos 60^\circ = 1$ m across, and the woman $x \cos 60^\circ = 0.5x$ m across:

$$\begin{aligned} 28 \times 4 \sin 60^\circ &= 10 \times 1 + 70 \times 0.5x \\ 96.99 \dots &= 10 + 35x \end{aligned}$$

so $x = 2.49$ m.

Question 7

[6 marks]

A rod AB of length 3 m and mass 5 kg rests with A on rough horizontal ground and touches a smooth peg at C , where $AC = 2.5$ m. The rod makes 30° with the ground and is in limiting equilibrium. Find the force from the peg and the coefficient of friction.

SOLUTION

The peg is smooth, so its force N is at right angles to the rod. Moments about A :

$$N \times 2.5 = 5g \times 1.5 \cos 30^\circ$$

$$N = 25.5 \text{ N}$$

N makes 30° with the vertical. Resolving:

$$F = N \sin 30^\circ = 12.73 \dots$$

$$R = 5g - N \cos 30^\circ = 26.95$$

Limiting, so $\mu = \frac{F}{R} = 0.472$.

Question 8

[5 marks]

Particle A , of mass 3 kg, rests on a rough plane inclined at 30° , with coefficient of friction 0.3. A string parallel to the plane joins it, over a pulley at the top, to particle B , of mass 2 kg, which hangs freely. Show that the system stays at rest, and find the friction on A .

SOLUTION

Up the plane, B pulls with $2g = 19.6$ N. Down the plane, the weight component is $3g \sin 30^\circ = 14.7$ N, and friction can give up to $0.3 \times 3g \cos 30^\circ = 7.638 \dots$ N.

19.6 is less than $14.7 + 7.638 \dots = 22.33 \dots$, so A can't be pulled up; and 14.7 is less than 19.6, so it can't slide down either. The system stays at rest.

With $T = 19.6$, the friction needed is $19.6 - 14.7 = 4.9$ N, acting **down** the plane, well within its limit.

Question 9

[7 marks]

Particle A , of mass 4 kg, lies on a rough horizontal table, with coefficient of friction 0.25. A string joins it, over a pulley at the edge, to particle B , of mass 2 kg, which hangs 0.5 m above the floor. The system is released from rest.

(a) Find the acceleration and the tension. [4]

(b) B hits the floor and does not rebound. Find how much further A slides, assuming it does not

reach the pulley. [3]

SOLUTION

(a) Friction on A is $0.25 \times 4g = g$:

$$2g - T = 2a \quad (B)$$

$$T - g = 4a \quad (A)$$

Adding: $g = 6a$, so $a = 1.63 \text{ m s}^{-2}$, and $T = 2(g - a) = 16.3 \text{ N}$.

(b) $v^2 = 2 \times \frac{g}{6} \times 0.5 = \frac{g}{6}$. Then only friction acts on A , decelerating it at $\frac{g}{4}$:

$$s = \frac{v^2}{2 \times \frac{g}{4}} = \frac{g/6}{g/2} = \frac{1}{3} \text{ m}$$

Exam style

Question 10

[9 marks]

Particle A , of mass 2 kg , rests on a rough plane inclined at α , where $\tan \alpha = \frac{3}{4}$, with coefficient of friction $\frac{1}{2}$. A string parallel to a line of greatest slope joins it, over a pulley at the top of the plane, to particle B , of mass 3 kg , which hangs 1 m above the ground. The system is released from rest.

- (a) Find the acceleration of the system. [3]
- (b) Find the tension in the string. [1]
- (c) Find the speed of B as it hits the ground. [2]
- (d) B does not rebound. Find how much further A moves up the plane, and state what happens to it next. [3]

SOLUTION

(a) For A : $R = 2g \cos \alpha = 1.6g$, so friction is $0.8g$; the weight component is $2g \sin \alpha = 1.2g$. B pulls with $3g$, more than $2g$, so the system moves:

$$3g - T = 3a \quad (B)$$

$$T - 1.2g - 0.8g = 2a \quad (A)$$

Adding: $g = 5a$, so $a = 1.96 \text{ m s}^{-2}$.

(b) $T = 3(g - a) = 23.5 \text{ N}$.

(c) $v^2 = 2 \times 1.96 \times 1 = 3.92$, so $v = 1.98 \text{ m s}^{-1}$.

(d) Now $1.2g$ and $0.8g$ both act down the plane on A , a deceleration of $\frac{2g}{2} = g$. So $s = \frac{3.92}{2g} = 0.2 \text{ m}$.

$\tan \alpha = 0.75$ is more than $\mu = 0.5$, so A then slides back down the plane.

Question 11

[8 marks]

A ladder AB , of length 4 m and mass 15 kg, rests with A on rough horizontal ground and B against a smooth vertical wall, at an angle α to the ground, where $\tan \alpha = \frac{12}{5}$. A painter of mass 75 kg stands on the ladder at C , where $AC = 3 \text{ m}$, and the ladder is on the point of slipping.

- Find the coefficient of friction. [5]
- Find the least coefficient of friction that would let the painter stand at the top of the ladder. [2]
- State how you have used the fact that the ladder is uniform. [1]

SOLUTION

(a) Vertically, $R = 90g$. Take moments about A : the ladder's weight acts $2 \cos \alpha$ across and the painter $3 \cos \alpha$ across, so the weights give $(15 \times 2 + 75 \times 3)g \cos \alpha = 255g \cos \alpha$:

$$4S \sin \alpha = 255g \cos \alpha$$

$$S = \frac{255g}{4 \tan \alpha} = 26.5625g$$

Horizontally $F = S$, and limiting, so $\mu = \frac{26.5625g}{90g} = 0.295$.

(b) With the painter at B , $4S \sin \alpha = (30 + 300)g \cos \alpha$, so

$$S = \frac{330g}{4 \tan \alpha} = 34.375g$$

$$\mu \geq \frac{34.375}{90} = 0.382$$

(c) Uniform means the ladder's weight acts at its midpoint, 2 m from A .

Question 12

[8 marks]

A ladder of weight W rests with its foot on rough horizontal ground and its top against a rough vertical wall. The coefficient of friction is $\frac{1}{2}$ at both. The ladder is on the point of slipping, at an angle θ to the ground.

- Show that the normal reaction at the ground is $\frac{4}{5}W$. [4]
- Show that $\tan \theta = \frac{3}{4}$. [4]

SOLUTION

(a) On the point of slipping, friction is limiting at both ends: at the foot $\frac{1}{2}R$ towards the wall, and at the top $\frac{1}{2}S$ **up** the wall, resisting the top sliding down. Resolving:

$$\begin{aligned}S &= \frac{1}{2}R \\ R + \frac{1}{2}S &= W\end{aligned}$$

so $R + \frac{1}{4}R = W$ and $R = \frac{4}{5}W$.

(b) $S = \frac{2}{5}W$. Let the ladder's length be $2l$. Moments about the foot: S and the wall's friction act at the top, which is $2l \sin \theta$ up and $2l \cos \theta$ across:

$$\begin{aligned}2Sl \sin \theta + Sl \cos \theta &= Wl \cos \theta \\ \frac{4}{5} \tan \theta + \frac{2}{5} &= 1\end{aligned}$$

putting $S = \frac{2}{5}W$ and dividing by $Wl \cos \theta$. So $\tan \theta = \frac{3}{4}$.