

YEAR 1 PURE · CHAPTER 15

Vectors

Specification: Edexcel Pure 10 — Vectors · AQA section J

KEY FACTS — THIS CHAPTER

- A **vector** has magnitude *and* direction; in 2D it's written $a\mathbf{i} + b\mathbf{j}$ or as a column $\begin{pmatrix} a \\ b \end{pmatrix}$.
- **Magnitude** (length): $\left| \begin{pmatrix} a \\ b \end{pmatrix} \right| = \sqrt{a^2 + b^2}$ — Pythagoras on the components.
- **Add/subtract** component by component; **multiply** by a scalar multiplies each component.
- **Displacement** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ (position of B minus position of A).
- **Parallel** vectors are scalar multiples of each other: $\mathbf{u} = k\mathbf{v}$.
- A point dividing AB in ratio $m : n$ has position vector $\mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a})$.

A number tells you *how much*; a vector tells you *how much and which way*. Vectors package a size and a direction into one object, which makes journeys, forces and displacements clean to calculate with.

1. Vectors, components and magnitude

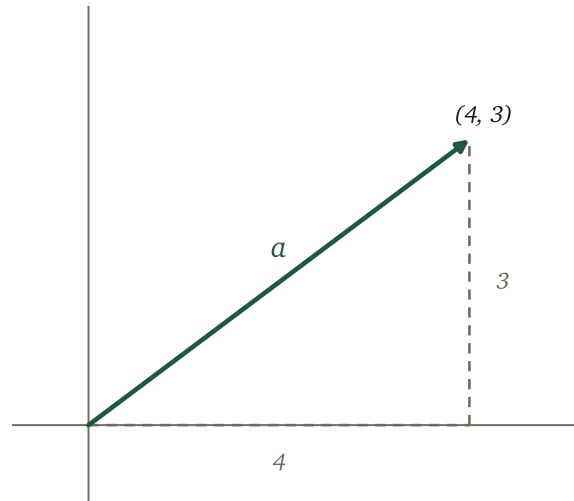
A 2D vector is described by how far it goes across and up: its **components**.

KEY RESULT

The vector $a\mathbf{i} + b\mathbf{j} = \begin{pmatrix} a \\ b \end{pmatrix}$ goes a across and b up. Its **magnitude** (length) is

$$\left| \begin{pmatrix} a \\ b \end{pmatrix} \right| = \sqrt{a^2 + b^2}$$

— just Pythagoras on the two components.



$a = 4\mathbf{i} + 3\mathbf{j}$: 4 across, 3 up, length $\sqrt{4^2 + 3^2} = 5$.

WORKED EXAMPLE 15.1 · MAGNITUDE OF A VECTOR

Find the magnitude of $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$.

Apply Pythagoras to the components (the signs don't matter once squared):

$$|\mathbf{v}| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

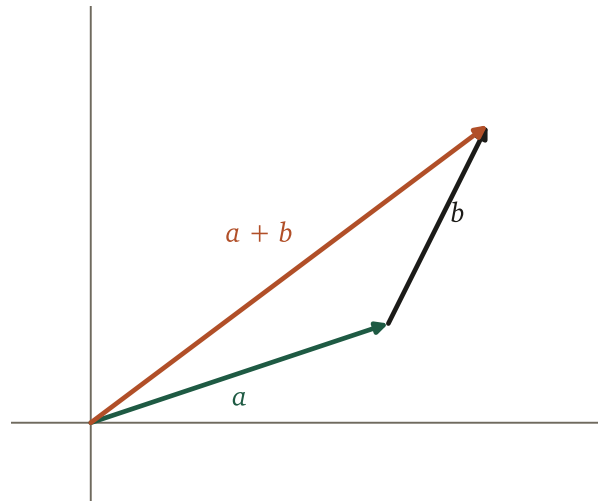
2. Vector arithmetic

Vectors add, subtract and scale exactly as you'd hope — one component at a time.

KEY RESULT

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}, \quad k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

Geometrically, **addition is tip-to-tail**: draw the first vector, start the second at its tip, and the **resultant** runs from the very start to the very end.



Follow $\mathbf{a}$ then $\mathbf{b}$; the resultant $\mathbf{a} + \mathbf{b}$ closes the triangle.

WORKED EXAMPLE 15.2 · COMBINING VECTORS

Given $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, find $2\mathbf{a} - \mathbf{b}$.

1. Scale $\mathbf{a}$ by 2:

$$2\mathbf{a} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

2. Subtract $\mathbf{b}$ component by component:

$$2\mathbf{a} - \mathbf{b} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

3. Position and displacement vectors

The **position vector** of a point A is $\overrightarrow{OA} = \mathbf{a}$, the vector from the origin to A . The vector *between* two points comes from subtracting their positions.

KEY RESULT

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} \quad (\text{position of } B \text{ minus position of } A)$$

The distance AB is then $|\overrightarrow{AB}|$.

WORKED EXAMPLE 15.3 · DISPLACEMENT AND DISTANCE

Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. Find $\overrightarrow{AB}$ and the distance AB .

1. Displacement is “end minus start”:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

2. Distance is its magnitude:

$$AB = \sqrt{3^2 + 4^2} = 5$$

COMMON MISTAKE

$\overrightarrow{AB}$ is $\mathbf{b} - \mathbf{a}$, not $\mathbf{a} - \mathbf{b}$ — *end minus start*. Getting it backwards gives $\overrightarrow{BA}$, the same length but the opposite direction.

4. Ratios and geometric problems

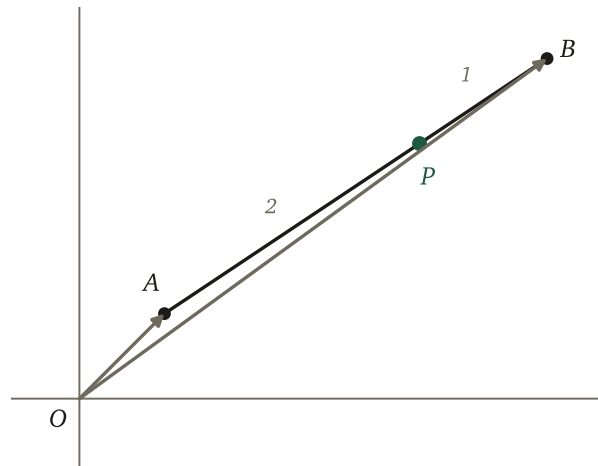
Vectors shine at “prove these points are collinear” and “find the point dividing this line in a given ratio” — the kind of geometry that’s fiddly with coordinates alone.

PARALLEL, AND DIVIDING A LINE

- Two vectors are **parallel** exactly when one is a scalar multiple of the other: $\mathbf{u} = k\mathbf{v}$. (If they also share a point, that makes the points **collinear**.)
- The point P dividing AB in the ratio $m : n$ has position vector

$$\overrightarrow{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a}).$$

(With $m = n$ this is the **midpoint**, $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.)



P divides AB in the ratio 2 : 1, two-thirds of the way from A to B.

WORKED EXAMPLE 15.4 · DIVIDING A LINE IN A RATIO

Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$. Find the position vector of the point P that divides AB in the ratio 2 : 1.

1. P is $\frac{2}{2+1} = \frac{2}{3}$ of the way from A to B :

$$\overrightarrow{OP} = \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a})$$

2. Compute $\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$, so $\frac{2}{3}(\mathbf{b} - \mathbf{a}) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$:

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

WORKED EXAMPLE 15.5 · SHOWING POINTS ARE COLLINEAR

Show that $A(1, 2)$, $B(3, 5)$ and $C(7, 11)$ are collinear.

- Find two displacement vectors from A :

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

- $\overrightarrow{AC} = 3\overrightarrow{AB}$, so the two vectors are parallel — and since they share the point A , the three points lie on one straight line. ■

COMMON MISTAKE

Parallel vectors alone don't prove collinearity: you also need a **shared point**. $\overrightarrow{AB} \parallel \overrightarrow{CD}$ just means two *separate* lines are parallel; it's $\overrightarrow{AB} \parallel \overrightarrow{AC}$ (same starting point A) that forces A, B, C onto one line.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

Find the magnitude of each vector:

(a) $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$

(b) $3\mathbf{i} - 4\mathbf{j}$

SOLUTION

(a) $\sqrt{6^2 + 8^2} = \sqrt{100} = 10.$

(b) $\sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$

Question 2

[3 marks]

Given $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, find:

(a) $\mathbf{a} + \mathbf{b}$

(b) $3\mathbf{a}$

(c) $\mathbf{a} - \mathbf{b}$

SOLUTION

(a) $\begin{pmatrix} 2+3 \\ -1+4 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$

(b) $\begin{pmatrix} 6 \\ -3 \end{pmatrix}.$

(c) $\begin{pmatrix} 2-3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}.$

Question 3

[3 marks]

Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$. Find $\overrightarrow{AB}$ and the distance AB .

SOLUTION

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \text{ and } AB = \sqrt{6^2 + 8^2} = 10.$$

Question 4

[2 marks]

Show that $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$ are parallel.

SOLUTION

$$\begin{pmatrix} 8 \\ 12 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ — one is a scalar multiple of the other, so they are parallel.}$$

Standard**Question 5**

[3 marks]

Find a unit vector in the direction of $\mathbf{v} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

SOLUTION

The magnitude is $|\mathbf{v}| = \sqrt{5^2 + (-12)^2} = 13$. A unit vector divides by the magnitude:

$$\hat{\mathbf{v}} = \frac{1}{13} \begin{pmatrix} 5 \\ -12 \end{pmatrix} = \begin{pmatrix} 5/13 \\ -12/13 \end{pmatrix}$$

Question 6

[3 marks]

Given $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, find $2\mathbf{a} + 3\mathbf{b}$.

SOLUTION

$$2\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 12 \\ -3 \end{pmatrix} = \begin{pmatrix} 14 \\ 1 \end{pmatrix}$$

Question 7

[4 marks]

The vector $\begin{pmatrix} 6 \\ p \end{pmatrix}$ is parallel to $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Find the value of p .

SOLUTION

For parallel vectors, $\begin{pmatrix} 6 \\ p \end{pmatrix} = k \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. The first components give $6 = 2k$, so $k = 3$. Then $p = 5k = 15$.

Question 8

[4 marks]

Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$. Find the position vector of the midpoint M of AB .

SOLUTION

$$\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2} \begin{pmatrix} 12 \\ 10 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

Exam style**Question 9**

[5 marks]

Points A , B and C have position vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 10 \end{pmatrix}$.

- (a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]
 (b) Hence show that A , B and C are collinear, and state the ratio $AB : BC$. [3]

SOLUTION

(a) $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$.

(b) $\overrightarrow{AC} = 3\overrightarrow{AB}$, so the vectors are parallel and share the point A — hence A , B , C are collinear. Since $\overrightarrow{AC} = 3\overrightarrow{AB}$, B is one-third of the way along, so $AB : BC = 1 : 2$.

Question 10

[5 marks]

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$. The point P divides AB in the ratio $1 : 2$.

- (a) Find $\overrightarrow{OP}$. [3]
 (b) Find the exact distance OP . [2]

SOLUTION

(a) P is $\frac{1}{3}$ of the way from A to B :

$$\overrightarrow{OP} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

(b) $OP = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$.

Question 11

[5 marks]

$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$. Point C is such that $OACB$ is a parallelogram (with OA and OB as adjacent sides). Find the position vector of C .

SOLUTION

In parallelogram $OACB$, $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$ (the diagonal is the sum of the two sides):

$$\overrightarrow{OC} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

Question 12

[5 marks]

The vector $\mathbf{c} = \begin{pmatrix} k \\ k+1 \end{pmatrix}$ has magnitude 5. Find the possible values of k .

SOLUTION

Set the magnitude squared equal to 25:

$$k^2 + (k+1)^2 = 25$$

Expand and tidy: $k^2 + k^2 + 2k + 1 = 25$, so $2k^2 + 2k - 24 = 0$, i.e. $k^2 + k - 12 = 0$.

Factorise: $(k+4)(k-3) = 0$, so $k = 3$ or $k = -4$.

(Check: $k = 3$ gives $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $k = -4$ gives $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ — both of magnitude 5.)