

YEAR 1 PURE • CHAPTER 8

Triangle Geometry

Specification: Edexcel Pure 5 — Trigonometry • AQA section E

KEY FACTS — THIS CHAPTER

- **Labelling:** side a is opposite angle A , side b opposite B , side c opposite C .
- **Right-angled triangles:** SOH-CAH-TOA — $\sin = \frac{\text{opp}}{\text{hyp}}$, $\cos = \frac{\text{adj}}{\text{hyp}}$, $\tan = \frac{\text{opp}}{\text{adj}}$.
- **Sine rule:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ — use it with a side and its opposite angle.
- **Cosine rule:** $a^2 = b^2 + c^2 - 2bc \cos A$ — use it for SAS (find a side) or SSS (find an angle).
- **Area:** $\frac{1}{2}ab \sin C$ — two sides and the angle *between* them.
- The sine rule can be **ambiguous**: $\sin \theta = \sin(180^\circ - \theta)$, so a “find the angle” sine-rule problem can have two answers.

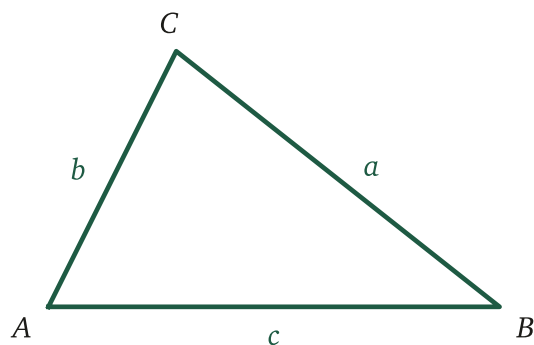
Once a triangle stops being right-angled, SOH-CAH-TOA alone can't finish the job. The sine and cosine rules pick up where it leaves off — between them they solve *any* triangle, given enough information.

1. Labelling and right-angled triangles

First, the convention that makes every formula in this chapter readable: each side is named by the angle it **faces**.

KEY RESULT

In triangle ABC , side a is opposite angle A , side b opposite B , and side c opposite C .



side *a* faces angle *A* • side *b* faces *B* • side *c* faces *C*

Each side faces the angle of the same letter — the key to reading the rules.

For a **right-angled** triangle you already have everything you need: SOH-CAH-TOA relates an angle to the opposite, adjacent and hypotenuse. Keep it in mind — the sine and cosine rules are both *derived* from it.

WORKED EXAMPLE 8.1 • RIGHT-ANGLED REFRESHER

A right-angled triangle has hypotenuse 10 and one angle of 35° . Find the side opposite the 35° angle.

The opposite side and the hypotenuse call for sine (SOH):

$$\sin 35^\circ = \frac{\text{opp}}{10} \implies \text{opp} = 10 \sin 35^\circ = 5.74 \text{ (3 s.f.)}$$

2. The sine rule

The sine rule links each side to the **sine of its opposite angle**. Reach for it whenever you have (or want) a side paired with the angle across from it.

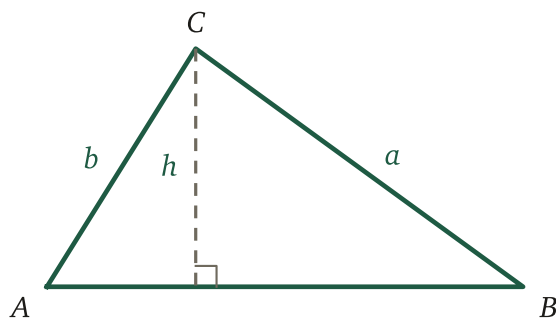
KEY RESULT

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Turn it upside down — $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ — when you're finding an **angle**.

WHY IT WORKS

Drop a perpendicular of height h from C onto AB . It splits the triangle into two right-angled triangles, and in each one h is the opposite side: $h = b \sin A$ from the left, $h = a \sin B$ from the right. Setting them equal, $b \sin A = a \sin B$, which rearranges to $\frac{a}{\sin A} = \frac{b}{\sin B}$.



$$h = b \sin A = a \sin B$$

The shared height h gives $b \sin A = a \sin B$ in one step.

WORKED EXAMPLE 8.2 · FINDING A SIDE WITH THE SINE RULE

In triangle ABC , $A = 50^\circ$, $B = 60^\circ$ and $a = 10$. Find b .

1. Pair each side with its opposite angle:

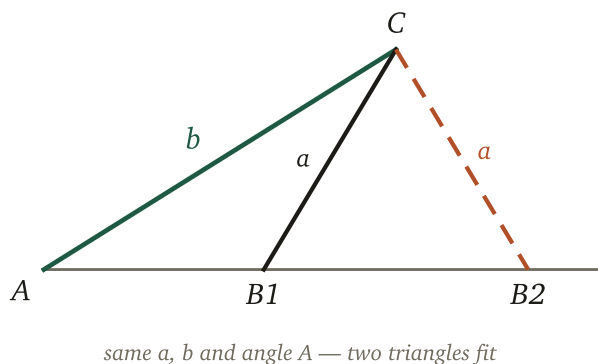
$$\frac{b}{\sin 60^\circ} = \frac{10}{\sin 50^\circ}$$

2. Rearrange and evaluate:

$$b = \frac{10 \sin 60^\circ}{\sin 50^\circ} = 11.3 \text{ (3 s.f.)}$$

The ambiguous case

Finding an *angle* with the sine rule comes with a catch: $\sin \theta$ and $\sin(180^\circ - \theta)$ are equal, so the calculator's answer might not be the only one.



With two sides and a non-included angle, the third vertex can land in two places.

WORKED EXAMPLE 8.3 · TWO POSSIBLE TRIANGLES

In triangle ABC , $A = 40^\circ$, $a = 9$ and $b = 12$. Find the two possible values of angle B .

1. Use the angle form of the sine rule:

$$\frac{\sin B}{12} = \frac{\sin 40^\circ}{9} \implies \sin B = \frac{12 \sin 40^\circ}{9} = 0.857$$

2. The calculator gives $B = 59.0^\circ$. But $\sin B = \sin(180^\circ - B)$, so $B = 121.0^\circ$ is also possible.
3. Check both fit ($A + B < 180^\circ$): $40 + 59 = 99^\circ \checkmark$ and $40 + 121 = 161^\circ \checkmark$ — both are valid triangles.

So $B = 59.0^\circ$ or $B = 121.0^\circ$.

COMMON MISTAKE

Don't trust the calculator's single answer when finding an angle. Always ask whether the obtuse partner $180^\circ - \theta$ also fits (i.e. whether the angles still sum to less than 180°). If it does, the problem has **two** solutions.

3. The cosine rule

When the sine rule can't start — because you don't have a side-and-its-opposite-angle pair — the cosine rule steps in. It needs either two sides and the angle between them, or all three sides.

KEY RESULT

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Here a is the side **opposite** the angle A , and b, c are the two sides that enclose A . Use it to **find a side** from SAS (two sides and the included angle). Rearranged for the angle,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

it **finds an angle** from SSS (all three sides). The letters just rotate — to find B instead, use

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and similarly for } C.$$

WORKED EXAMPLE 8.4 • FINDING A SIDE (SAS)

In triangle ABC , $b = 8$, $c = 5$ and the included angle $A = 60^\circ$. Find a .

1. Substitute into the cosine rule — a is opposite the known angle A :

$$a^2 = 8^2 + 5^2 - 2(8)(5) \cos 60^\circ$$

2. Evaluate, using $\cos 60^\circ = \frac{1}{2}$:

$$a^2 = 64 + 25 - 80 \times \frac{1}{2} = 89 - 40 = 49$$

So $a = 7$.

WORKED EXAMPLE 8.5 • FINDING AN ANGLE (SSS)

A triangle has sides 5, 7 and 8. Find the size of the angle opposite the side of length 7.

1. Put the wanted angle as A , so its opposite side is $a = 7$, with the other two $b = 5$ and $c = 8$. Rearrange the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 64 - 49}{2(5)(8)} = \frac{40}{80} = \frac{1}{2}$$

2. So $A = 60^\circ$.

COMMON MISTAKE

The angle in the cosine rule must be the one **opposite** the side you're finding (equivalently, the angle **between** the two known sides). In $a^2 = b^2 + c^2 - 2bc \cos A$, the angle A sits between sides b and c and faces side a — never one of the angles at the other corners.

4. Area of a triangle

The familiar $\frac{1}{2} \times \text{base} \times \text{height}$ becomes far more useful once you can write the height in terms of a side and an angle.

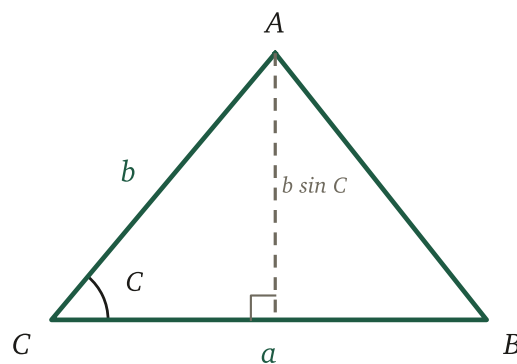
KEY RESULT

$$\text{Area} = \frac{1}{2}ab \sin C$$

— two sides and the angle **between** them. (Any pair of sides works, as long as you use the angle they enclose.)

WHY IT WORKS

Take base a . The perpendicular height onto it, dropped from the opposite vertex, is $b \sin C$ — the side b times the sine of the angle C it makes with the base. So $\text{Area} = \frac{1}{2} \times a \times b \sin C$.



$$\text{Area} = \frac{1}{2}ab \sin C$$

The height onto base a is $b \sin C$, so the area is $\frac{1}{2}ab \sin C$.

WORKED EXAMPLE 8.6 · AREA FROM TWO SIDES AND THE INCLUDED ANGLE

Find the area of the triangle with $a = 8$, $b = 5$ and included angle $C = 60^\circ$.

$$\text{Area} = \frac{1}{2}(8)(5) \sin 60^\circ = 20 \sin 60^\circ = 17.3 \text{ (3 s.f.)}$$

(This is the same triangle as in Example 8.4 — sides 5, 7, 8 — so its area is $10\sqrt{3}$ in exact form.)

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers. Give non-exact answers to 3 s.f.

Warm-up**Question 1**

[3 marks]

In triangle ABC , $A = 45^\circ$, $B = 65^\circ$ and $a = 12$. Find b .

SOLUTION

Sine rule, pairing each side with its opposite angle:

$$\frac{b}{\sin 65^\circ} = \frac{12}{\sin 45^\circ} \implies b = \frac{12 \sin 65^\circ}{\sin 45^\circ} = 15.4 \text{ (3 s.f.)}$$

Question 2

[3 marks]

In triangle ABC , $b = 6$, $c = 9$ and the included angle $A = 40^\circ$. Find a .

SOLUTION

Cosine rule, with a opposite the known angle A :

$$a^2 = 6^2 + 9^2 - 2(6)(9) \cos 40^\circ = 36 + 81 - 108 \cos 40^\circ = 34.267 \dots$$

So $a = \sqrt{34.267 \dots} = 5.85 \text{ (3 s.f.)}$.

Question 3

[3 marks]

Find the area of the triangle with sides $a = 7$ and $b = 10$ enclosing an angle $C = 55^\circ$.

SOLUTION

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{1}{2}(7)(10) \sin 55^\circ = 35 \sin 55^\circ = 28.7 \text{ (3 s.f.)}$$

Question 4

[2 marks]

A triangle has sides 4, 6 and 9. State, with a reason, whether the angle opposite the side of length 9 is acute or obtuse.

SOLUTION

By the cosine rule the cosine of that angle is

$$\cos \theta = \frac{4^2 + 6^2 - 9^2}{2(4)(6)} = \frac{16 + 36 - 81}{48} = \frac{-29}{48} < 0$$

A negative cosine means the angle is **obtuse**.

Standard**Question 5**

[4 marks]

In triangle ABC , $a = 10$, $b = 7$ and $c = 8$. Find angle A .

SOLUTION

A is opposite the longest side, so use the cosine rule rearranged:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{49 + 64 - 100}{2(7)(8)} = \frac{13}{112} = 0.11607 \dots$$

$$A = \cos^{-1}(0.11607 \dots) = 83.3^\circ \text{ (3 s.f.)}$$

Question 6

[4 marks]

In triangle ABC , $A = 35^\circ$, $a = 9$ and $b = 13$. Find the two possible values of angle B .

SOLUTION

Angle form of the sine rule:

$$\frac{\sin B}{13} = \frac{\sin 35^\circ}{9} \implies \sin B = \frac{13 \sin 35^\circ}{9} = 0.82849 \dots$$

The calculator gives $B = 55.9^\circ$. Its obtuse partner $B = 180^\circ - 55.9^\circ = 124.1^\circ$ also fits, since $35 + 124.1 = 159.1^\circ < 180^\circ$.

So $B = 55.9^\circ$ or $B = 124^\circ$ (3 s.f.).

Question 7

[5 marks]

In triangle ABC , $b = 11$, $c = 6$ and $A = 72^\circ$.

- (a) Find a . [3]
- (b) Find the area of the triangle. [2]

SOLUTION

(a) Cosine rule (SAS, angle A between sides b and c):

$$\begin{aligned} a^2 &= 11^2 + 6^2 - 2(11)(6) \cos 72^\circ \\ &= 121 + 36 - 132 \cos 72^\circ = 116.2 \dots \end{aligned}$$

$$a = \sqrt{116.2 \dots} = 10.8 \text{ (3 s.f.)}$$

(b) Area $= \frac{1}{2}bc \sin A = \frac{1}{2}(11)(6) \sin 72^\circ = 33 \sin 72^\circ = 31.4$ (3 s.f.).

Question 8

[5 marks]

In triangle ABC , $A = 48^\circ$, $B = 61^\circ$ and $b = 14$.

- (a) Find angle C . [1]
- (b) Find side c . [2]
- (c) Find the area of the triangle. [2]

SOLUTION

(a) Angles sum to 180° : $C = 180^\circ - 48^\circ - 61^\circ = 71^\circ$.

(b) Sine rule:

$$c = \frac{b \sin C}{\sin B} = \frac{14 \sin 71^\circ}{\sin 61^\circ} = 15.13 \dots = 15.1 \text{ (3 s.f.)}$$

(c) Two sides and the included angle — use b , c and angle A , keeping the unrounded c from part (b):

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}(14)(15.13 \dots) \sin 48^\circ = 78.7 \text{ (3 s.f.)}$$

Rounding c to 15.1 first would give 78.6: carry full accuracy through, and round only the final answer.

Question 9

[4 marks]

The area of triangle ABC is 40 cm^2 . Given $a = 12 \text{ cm}$ and $b = 9 \text{ cm}$, find the two possible values of the included angle C .

SOLUTION

$$\frac{1}{2}(12)(9) \sin C = 40 \implies 54 \sin C = 40 \implies \sin C = 0.7407 \dots$$

The calculator gives $C = 47.8^\circ$; the obtuse partner $C = 132.2^\circ$ also gives this area. So $C = 47.8^\circ$ or $C = 132^\circ$ (3 s.f.).

Exam style

Question 10

[6 marks]

A ship sails 8 km on a bearing of 030° from P to Q , then 11 km on a bearing of 100° from Q to R .

- (a) Show that angle $PQR = 110^\circ$. [2]
- (b) Find the distance PR . [3]
- (c) Find the area of triangle PQR . [1]

SOLUTION

(a) The bearing of P from Q is $030^\circ + 180^\circ = 210^\circ$. The angle between QP (210°) and QR (100°) is $210^\circ - 100^\circ = 110^\circ$. ✓

(b) Cosine rule with the 110° angle between $PQ = 8$ and $QR = 11$:

$$\begin{aligned} PR^2 &= 8^2 + 11^2 - 2(8)(11) \cos 110^\circ \\ &= 64 + 121 - 176 \cos 110^\circ = 245.19 \dots \end{aligned}$$

$$PR = 15.7 \text{ km (3 s.f.)}$$

$$(c) \text{ Area} = \frac{1}{2}(8)(11) \sin 110^\circ = 44 \sin 110^\circ = 41.3 \text{ km}^2 \text{ (3 s.f.)}.$$

Question 11

[5 marks]

In triangle ABC , $a = 9$, $b = 10$ and $c = 11$.

(a) Find the largest angle of the triangle. [3]

(b) Hence find the area of the triangle. [2]

SOLUTION

(a) The largest angle faces the longest side $c = 11$, so find C :

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{81 + 100 - 121}{2(9)(10)} = \frac{60}{180} = \frac{1}{3}$$

$$C = \cos^{-1}\left(\frac{1}{3}\right) = 70.5^\circ \text{ (3 s.f.)}$$

$$(b) \text{ Area} = \frac{1}{2}ab \sin C = \frac{1}{2}(9)(10) \sin 70.5^\circ = 45 \sin 70.5^\circ = 42.4 \text{ (3 s.f.)}.$$

Question 12

[6 marks]

Points A , B and C lie on a circle of radius 7, with centre O . The angle AOB (at the centre) is 80° .

(a) Find the length of the chord AB . [3]

(b) Find the area of triangle AOB . [1]

(c) Hence find the area of the minor segment cut off by the chord AB . [2]

SOLUTION

(a) Triangle AOB has $OA = OB = 7$ (radii) with the 80° angle between them. Cosine rule:

$$AB^2 = 7^2 + 7^2 - 2(7)(7) \cos 80^\circ = 98 - 98 \cos 80^\circ = 80.982 \dots$$

$$AB = 9.00 \text{ (3 s.f.)}$$

$$(b) \text{ Area of triangle } AOB = \frac{1}{2}(7)(7) \sin 80^\circ = 24.5 \sin 80^\circ = 24.1.$$

(c) The sector AOB has area $\frac{80}{360} \times \pi \times 7^2 = 34.2$. The segment is the sector minus the triangle:

$$34.2 - 24.1 = 10.1 \text{ (3 s.f.)}$$