

YEAR 1 PURE · CHAPTER 2

Quadratics

Specification: Edexcel Pure 2 — Algebra and functions · AQA section B

KEY FACTS — THIS CHAPTER

- A quadratic $y = ax^2 + bx + c$ is a **parabola**: opens upwards if $a > 0$, downwards if $a < 0$; y -intercept at $(0, c)$.
- Solve by **factorising** when you can, by the **formula** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when you can't.
- **Completing the square**: $y = a(x - p)^2 + q$ puts the turning point at (p, q) .
- The **discriminant** $\Delta = b^2 - 4ac$ counts real roots: two ($\Delta > 0$), one repeated ($\Delta = 0$), none ($\Delta < 0$).
- Solve **quadratic inequalities** by finding the roots, then reading the answer off a sketch.
- A **disguised quadratic** becomes a real one with a substitution like $u = x^2$ or $u = 2^x$.

Quadratics are the most-used objects in A-level maths — they turn up inside coordinate geometry, trigonometry, logarithms and mechanics, usually uninvited. This chapter is about knowing them from every angle: as graphs, as equations to solve, and as expressions to rewrite.

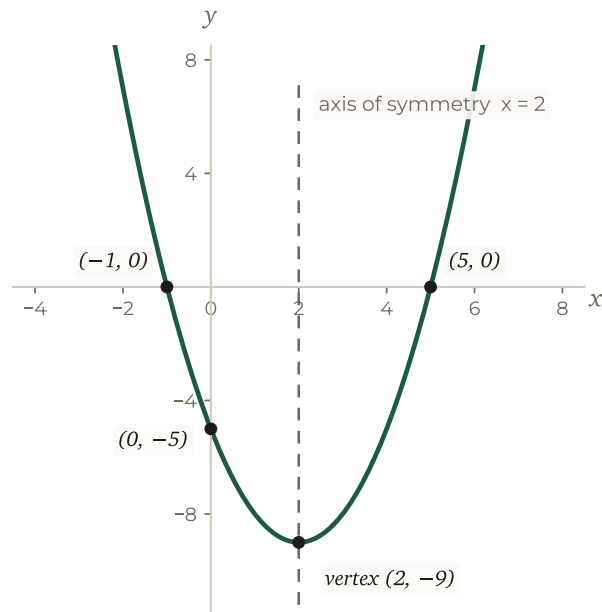
1. The shape of a quadratic

Every quadratic $y = ax^2 + bx + c$ draws the same curve — a **parabola** — and you can read its key features straight from the algebra:

ANATOMY OF A PARABOLA

- **Direction**: opens upwards if $a > 0$, downwards if $a < 0$.
- **y -intercept**: at $(0, c)$ — substitute $x = 0$.
- **Roots**: where $y = 0$, the crossings of the x -axis (if any).
- **Axis of symmetry**: the vertical line midway between the roots.
- **Vertex** (turning point): sits on the axis of symmetry — the minimum if the parabola

opens upwards, the maximum if it opens downwards.



Every feature of $y = x^2 - 4x - 5$, read straight from the algebra.

WORKED EXAMPLE 2.1 • READING OFF THE FEATURES

For $f(x) = x^2 - 4x - 5$, find the direction, the y -intercept, the roots, the axis of symmetry and the vertex.

1. $a = 1 > 0$, so the parabola opens **upwards**; the constant term gives the y -intercept $(0, -5)$.
2. Factorise to find the roots:

$$x^2 - 4x - 5 = (x - 5)(x + 1) = 0 \implies x = 5 \text{ or } x = -1$$

3. The axis of symmetry is midway between the roots:

$$x = \frac{5 + (-1)}{2} = 2$$

4. The vertex sits on that line — substitute $x = 2$:

$$f(2) = 4 - 8 - 5 = -9, \text{ so the vertex is } (2, -9)$$

2. Solving quadratic equations

Rearrange to $ax^2 + bx + c = 0$ first — always. Then factorise if the numbers cooperate, and reach for the formula when they don't.

WORKED EXAMPLE 2.2 • FACTORISING WITH A LEADING COEFFICIENT

Solve $2x^2 + 5x - 3 = 0$.

1. Look for a factorisation $(2x \quad)(x \quad)$ whose outer and inner terms combine to $+5x$:

$$2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

2. A product is zero only when one factor is zero:

$$2x - 1 = 0 \text{ or } x + 3 = 0 \implies x = \frac{1}{2} \text{ or } x = -3$$

THE QUADRATIC FORMULA

The solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It always works — it's completing the square done once, in general, so you never have to do it again.

WORKED EXAMPLE 2.3 • USING THE FORMULA, LEAVING AN EXACT ANSWER

Solve $3x^2 - 5x - 1 = 0$, giving your answers in exact form.

1. Identify $a = 3$, $b = -5$, $c = -1$ and substitute carefully:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{25 + 12}}{6}$$

2. Simplify under the root:

$$x = \frac{5 \pm \sqrt{37}}{6}$$

$\sqrt{37}$ has no square factor, so this is the final exact answer — leave it as a surd unless the question asks for decimals.

COMMON MISTAKE

Sign slips in the formula cost more marks than anything else here: $-b$ when b is already negative gives $+5$, and $-4ac$ with $c < 0$ turns into a **plus**. Write the substitution line in full before simplifying anything.

3. Completing the square

Completing the square rewrites a quadratic so that x appears only once — and in that form, the graph's vertex is sitting in plain sight.

KEY RESULT

$$y = a(x - p)^2 + q \text{ has its turning point at } (p, q)$$

A square is never negative, so $a(x - p)^2 + q$ is smallest (or largest, if $a < 0$) exactly when the square is zero — at $x = p$, where $y = q$.

WORKED EXAMPLE 2.4 · COMPLETING THE SQUARE

Write $x^2 + 6x + 5$ in the form $(x + p)^2 + q$ and state the turning point of $y = x^2 + 6x + 5$.

1. Halve the coefficient of x , square it inside, correct it outside:

$$x^2 + 6x + 5 = (x + 3)^2 - 9 + 5 = (x + 3)^2 - 4$$

2. Read off the turning point: the square vanishes at $x = -3$, giving the minimum point $(-3, -4)$.

WORKED EXAMPLE 2.5 · COMPLETING THE SQUARE WITH A LEADING COEFFICIENT

Write $2x^2 - 8x + 11$ in the form $a(x - p)^2 + q$, and hence find the minimum value of the expression.

1. Take the factor of 2 out of the x -terms only:

$$2x^2 - 8x + 11 = 2(x^2 - 4x) + 11$$

2. Complete the square inside the bracket:

$$2((x - 2)^2 - 4) + 11$$

3. Multiply out the correction and tidy:

$$2(x - 2)^2 - 8 + 11 = 2(x - 2)^2 + 3$$

The square is never negative, so the minimum value is 3, at $x = 2$.

COMMON MISTAKE

In step 3 the correction term gets multiplied by a on the way out of the bracket: the -4 becomes -8 , not -4 . Forgetting that factor is the classic completing-the-square error when $a \neq 1$.

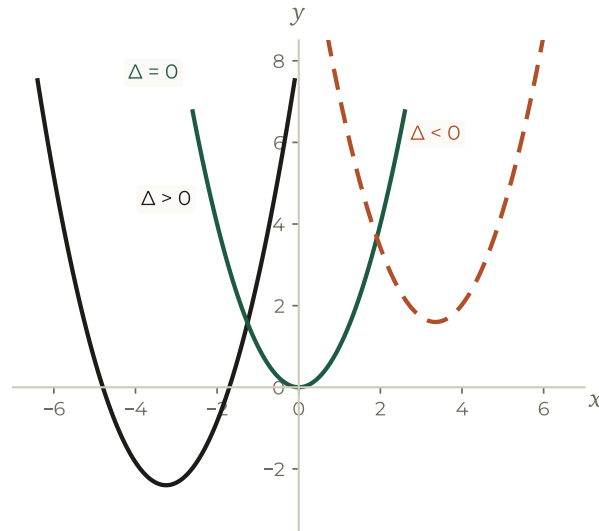
4. The discriminant

Inside the quadratic formula sits $b^2 - 4ac$, the **discriminant** Δ . Before any square root is taken, its sign already tells you how many real solutions exist — which is often all a question wants to know.

KEY RESULT

For $ax^2 + bx + c = 0$:

- $\Delta = b^2 - 4ac > 0$ — **two** distinct real roots: the graph crosses the x -axis twice
- $\Delta = 0$ — **one repeated** root: the graph touches the axis
- $\Delta < 0$ — **no** real roots: the graph misses the axis entirely



The three discriminant cases — crossing, touching, missing.

WORKED EXAMPLE 2.6 • USING THE DISCRIMINANT TO FIND AN UNKNOWN

The equation $x^2 + kx + 9 = 0$ has a repeated root. Find the possible values of k .

1. A repeated root means $\Delta = 0$, with $a = 1$, $b = k$, $c = 9$:

$$k^2 - 4(1)(9) = 0$$

2. Solve for k :

$$k^2 = 36 \implies k = 6 \text{ or } k = -6$$

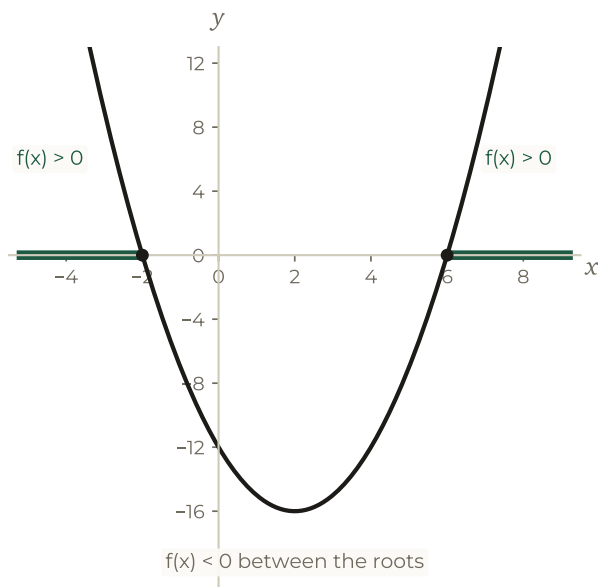
(Sense check: $x^2 + 6x + 9 = (x + 3)^2$ and $x^2 - 6x + 9 = (x - 3)^2$ — both perfect squares, as a repeated root demands.)

5. Quadratic inequalities

To solve something like $x^2 - 4x - 12 > 0$, don't push symbols around — solve the matching **equation**, then let a sketch answer the question.

THE SKETCH METHOD

1. Rearrange so one side is zero, and find the roots of the matching equation.
2. Sketch the parabola through those roots (direction from the sign of a).
3. Read off where the graph is above (> 0) or below (< 0) the axis: an upward parabola is **below** between its roots, **above** outside them.



$x^2 - 4x - 12 > 0$ exactly where the curve sits above the axis: outside the roots.

WORKED EXAMPLE 2.7 • SOLVING A QUADRATIC INEQUALITY

Solve $x^2 - 4x - 12 > 0$.

1. Solve the equation first:

$$x^2 - 4x - 12 = (x - 6)(x + 2) = 0 \implies x = 6 \text{ or } x = -2$$

2. $a = 1 > 0$, so the parabola opens upwards: below the axis between the roots, above it outside them.
3. We want > 0 — the outside regions:

$$x < -2 \text{ or } x > 6$$

COMMON MISTAKE

An answer like $6 < x < -2$ is impossible — no number is both greater than 6 and less than -2 . “Outside the roots” is always written as **two separate inequalities** joined by “or”. Save the single sandwich $-2 < x < 6$ for the between case.

6. Disguised quadratics

Some equations are quadratics wearing a costume: the variable appears twice, once as the square of the other appearance. A substitution makes the costume fall off.

KEY RESULT

If an equation contains $\square$ and $\square^2$ for some block $\square$ (like x^2 and x^4 , or 2^x and 2^{2x} , or $\sqrt{x}$ and x), substitute $u = \square$ to get an ordinary quadratic in u — then **convert back** to x at the end.

WORKED EXAMPLE 2.8 · A DISGUISED QUADRATIC

Solve $x^4 - 5x^2 + 4 = 0$.

1. Spot the disguise: $x^4 = (x^2)^2$, so let $u = x^2$:

$$u^2 - 5u + 4 = 0$$

2. Factorise and solve for u :

$$(u - 1)(u - 4) = 0 \implies u = 1 \text{ or } u = 4$$

3. Convert back — each positive u gives **two** values of x :

$$x^2 = 1 \implies x = \pm 1 \qquad x^2 = 4 \implies x = \pm 2$$

Four solutions: $x = -2, -1, 1, 2$.

COMMON MISTAKE

Stopping at $u = 1, 4$ loses half the marks — the question asked about x . And when converting back, check each value makes sense: $u = 2^x$ can never be negative, so a negative u root gets **discarded**, not converted.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

Solve by factorising:

(a) $x^2 - 9x + 20 = 0$

(b) $x^2 + 3x - 28 = 0$

SOLUTION

(a) Two numbers multiplying to 20 and adding to -9 : -4 and -5 .

$$(x - 4)(x - 5) = 0 \implies x = 4 \text{ or } x = 5$$

(b) Two numbers multiplying to -28 and adding to 3 : 7 and -4 .

$$(x + 7)(x - 4) = 0 \implies x = -7 \text{ or } x = 4$$

Question 2

[3 marks]

For the curve $y = x^2 - 2x - 8$, write down:

(a) the y -intercept,

(b) the roots,

(c) the equation of the axis of symmetry.

SOLUTION

(a) Substitute $x = 0$: the y -intercept is $(0, -8)$.

(b) Factorise: $x^2 - 2x - 8 = (x - 4)(x + 2)$, so the roots are $x = 4$ and $x = -2$.

(c) Midway between the roots: $x = \frac{4 + (-2)}{2} = 1$.

Question 3

[3 marks]

Write $x^2 + 8x + 3$ in the form $(x + p)^2 + q$, and state the coordinates of the turning point of $y = x^2 + 8x + 3$.

SOLUTION

Halve the 8, square it inside, correct outside:

$$x^2 + 8x + 3 = (x + 4)^2 - 16 + 3 = (x + 4)^2 - 13$$

The square vanishes at $x = -4$, so the turning point (a minimum) is $(-4, -13)$.

Question 4

[2 marks]

Calculate the discriminant of $3x^2 + 2x + 5 = 0$ and state the number of real roots.

SOLUTION

$$\Delta = b^2 - 4ac = 2^2 - 4(3)(5) = 4 - 60 = -56$$

$\Delta < 0$, so the equation has **no real roots** — the parabola never reaches the x -axis.

Standard**Question 5**

[3 marks]

Solve $2x^2 - 3x - 4 = 0$, giving your answers in exact form.

SOLUTION

It doesn't factorise (check: $\Delta = 9 + 32 = 41$, not a square), so use the formula with $a = 2$, $b = -3$, $c = -4$:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)} = \frac{3 \pm \sqrt{9 + 32}}{4}$$

$$x = \frac{3 \pm \sqrt{41}}{4}$$

Question 6

[4 marks]

Write $3x^2 + 12x - 7$ in the form $a(x+p)^2 + q$, and hence write down the minimum value of $3x^2 + 12x - 7$ and the value of x at which it occurs.

SOLUTION

1. Take the 3 out of the x -terms only:

$$3x^2 + 12x - 7 = 3(x^2 + 4x) - 7$$

2. Complete the square inside the bracket:

$$3((x+2)^2 - 4) - 7$$

3. Multiply the correction out and tidy:

$$3(x+2)^2 - 12 - 7 = 3(x+2)^2 - 19$$

The minimum value is -19 , at $x = -2$.

Question 7

[3 marks]

The equation $x^2 + kx + 25 = 0$ has a repeated root. Find the possible values of k .

SOLUTION

A repeated root means the discriminant is zero:

$$k^2 - 4(1)(25) = 0 \implies k^2 = 100 \implies k = \pm 10$$

(Check: $k = 10$ gives $(x+5)^2$ and $k = -10$ gives $(x-5)^2$.)

Question 8

[4 marks]

Solve the inequality $3x^2 - 5x - 2 < 0$.

SOLUTION

1. Solve the matching equation by factorising:

$$3x^2 - 5x - 2 = (3x + 1)(x - 2) = 0 \implies x = -\frac{1}{3} \text{ or } x = 2$$

2. $a = 3 > 0$: the parabola opens upwards, so it is **below** the axis between its roots.
3. We want < 0 , the between region:

$$-\frac{1}{3} < x < 2$$

Question 9

[4 marks]

Solve $2^{2x} - 9 \times 2^x + 8 = 0$.

SOLUTION

1. Spot the disguise: $2^{2x} = (2^x)^2$, so let $u = 2^x$:

$$u^2 - 9u + 8 = 0$$

2. Factorise: $(u - 1)(u - 8) = 0$, so $u = 1$ or $u = 8$.
3. Convert back to x :

$$2^x = 1 \implies x = 0 \qquad 2^x = 8 \implies x = 3$$

Exam style**Question 10**

[4 marks]

$$f(x) = x^2 - 6x + 11$$

- (a) Write $f(x)$ in the form $(x - p)^2 + q$. **[2]**
- (b) Hence explain why $f(x) > 0$ for all real values of x . **[1]**
- (c) Write down the coordinates of the minimum point of $y = f(x)$. **[1]**

SOLUTION

(a) $f(x) = (x - 3)^2 - 9 + 11 = (x - 3)^2 + 2$.

(b) $(x - 3)^2 \geq 0$ for every real x , so $f(x) \geq 0 + 2 = 2 > 0$ — the expression can never reach zero, let alone go below it.

(c) The minimum is where the square vanishes: $(3, 2)$.

Question 11

[5 marks]

The curve $y = x^2 + 4x + c$, where c is a constant.

(a) Find the value of c for which the curve touches the x -axis at exactly one point. [2]

(b) For that value of c , state the x -coordinate of the point of contact. [1]

(c) Find the range of values of c for which the curve does not meet the x -axis at all. [2]

SOLUTION

(a) Touching once means $\Delta = 0$:

$$4^2 - 4(1)(c) = 0 \implies 16 = 4c \implies c = 4$$

(b) With $c = 4$: $x^2 + 4x + 4 = (x + 2)^2$, so the contact point is at $x = -2$.

(c) No meeting means $\Delta < 0$:

$$16 - 4c < 0 \implies c > 4$$

Question 12

[5 marks]

The line $y = kx - 2$ is a tangent to the curve $y = x^2 + 3x + 7$. Find the two possible values of k .

SOLUTION

1. At an intersection the y -values agree, so set the expressions equal:

$$x^2 + 3x + 7 = kx - 2 \implies x^2 + (3 - k)x + 9 = 0$$

2. A tangent touches exactly once, so this quadratic has a repeated root — discriminant zero:

$$(3 - k)^2 - 4(1)(9) = 0$$

3. Solve:

$$(3 - k)^2 = 36 \implies 3 - k = \pm 6 \implies k = -3 \text{ or } k = 9$$

(Check $k = 9$: $x^2 - 6x + 9 = (x - 3)^2$ ✓ — a repeated root, as a tangent demands.)