

YEAR 1 PURE · CHAPTER 6

Polynomials and Algebraic Methods

Specification: Edexcel Pure 2 — Algebra and functions · AQA section B

KEY FACTS — THIS CHAPTER

- A **polynomial** is a sum of whole-number powers of x ; its **degree** is the highest power.
- **Polynomial division** has two standard write-ups: **long division** and the **grid (box) method** — they always agree, so use whichever you find clearer.
- **Factor theorem:** if $f(k) = 0$ then $(x - k)$ is a factor of $f(x)$ — and the reverse holds too.
- **Remainder theorem:** dividing $f(x)$ by $(x - k)$ leaves remainder $f(k)$.
- To **factorise a cubic**: find one root by trial (factor theorem), divide it out, then factorise the quadratic left behind.
- A factorised polynomial **crosses** the x -axis at each single root and **touches** at a repeated one.

Quadratics you can crack with the formula. Cubics and beyond have no friendly formula — so we factorise them, and the key that unlocks the first factor is the factor theorem. This chapter is the toolkit for taking a cubic apart.

1. Dividing polynomials

Dividing one polynomial by another is the engine of this chapter. There are two ways to lay the work out — **long division** and the **grid method**. They give the same answer every time; learn both and use whichever you trust under pressure.

Method 1 — long division

This mirrors the long division of numbers you met years ago: **divide, multiply, subtract, bring down, repeat**.

WORKED EXAMPLE 6.1 · LONG DIVISION

Divide $x^3 - 2x^2 - 5x + 6$ by $(x - 1)$.

Set it out like a long division, keeping like powers in the same column. Each step, divide the leading term to get the next part of the quotient (written on top), write the multiple of $(x - 1)$ underneath, and subtract — the working down the right shows exactly what each row is:

$$\begin{array}{r}
 \begin{array}{r}
 x^2 \quad -x \quad -6 \\
 (x-1) \overline{) \begin{array}{r} x^3 \quad -2x^2 \quad -5x \quad +6 \\
 -(x^3 \quad -x^2) \\
 \hline \quad -x^2 \quad -5x \quad +6 \\
 -(-x^2 \quad +x) \\
 \hline \qquad -6x \quad +6 \\
 -(-6x \quad +6) \\
 \hline \qquad \qquad 0
 \end{array}
 \end{array}
 \end{array}
 \begin{array}{l}
 x^2(x-1) = x^3 - x^2 \\
 -x(x-1) = -x^2 + x \\
 -6(x-1) = -6x + 6
 \end{array}$$

Each subtracted row is a multiple of $(x - 1)$; the quotient builds along the top, leaving remainder 0.

So $x^3 \div x = x^2$ (subtract $x^2(x - 1)$), then $-x^2 \div x = -x$ (subtract $-x(x - 1)$), then $-6x \div x = -6$ (subtract $-6(x - 1)$) — nothing left. The quotient sits along the top and the remainder is 0:

$$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$$

COMMON MISTAKE

Keep a slot for **every** power, even missing ones. Dividing $x^3 - 7x + 6$, write it as $x^3 + 0x^2 - 7x + 6$ — the invisible $0x^2$ keeps your columns lined up. Skipping it is the most common division error.

Method 2 — the grid method

Here you fill a **multiplication grid**. The divisor goes down the left, the quotient sits along the top (the part you're finding), and you fill each cell so that the columns add up to the polynomial you started with.

WORKED EXAMPLE 6.2 · THE GRID METHOD

Divide $x^3 - 2x^2 - 5x + 6$ by $(x - 1)$ using a grid.

1. Set up the grid with $(x - 1)$ down the left and the quotient (to be found) along the top. Fill it so each column matches a term of the dividend, working left to right:

$\times$	x^2	$-x$	-6
x	x^3	$-x^2$	$-6x$
-1	$-x^2$	$+x$	$+6$

2. Read the columns against $x^3 - 2x^2 - 5x + 6$:

- x^3 column: x^3 ✓ — so the first quotient term is x^2 . - x^2 column: $-x^2 - x^2 = -2x^2$ ✓ — the next term is $-x$. - x column: $-6x + x = -5x$ ✓ — the last term is -6 . - constant: $+6$ ✓.

The quotient is the top row: $x^2 - x - 6$ — exactly what long division gave.

WHY IT WORKS

The grid is just the expansion of $(x - 1)(x^2 - x - 6)$ laid out in a table: every cell is one product, and adding the cells down each column collects the like terms. Matching those column totals to the original polynomial is the same as saying the factors multiply back to it.

The grid copes just as happily with a non-monic divisor like $(2x + 3)$ — put $2x$ and $+3$ down the left and fill in the same way.

2. The factor and remainder theorems

Division is the workhorse, but there's a shortcut for the question “is $(x - k)$ a factor?” — you don't need to divide at all.

THE REMAINDER AND FACTOR THEOREMS

When $f(x)$ is divided by $(x - k)$, the remainder is exactly $f(k)$.

So if $f(k) = 0$, the remainder is zero and $(x - k)$ is a **factor**. This is the **factor theorem**, and it runs both ways:

$$(x - k) \text{ is a factor of } f(x) \iff f(k) = 0$$

WHY IT WORKS

Write the division as $f(x) = (x - k)Q(x) + R$, where R is the (constant) remainder. Substituting $x = k$ kills the first term — $(k - k) = 0$ — leaving $f(k) = R$. So the remainder is $f(k)$, and it vanishes exactly when $(x - k)$ divides $f(x)$ cleanly.

WORKED EXAMPLE 6.3 · TESTING FOR FACTORS

Show that $(x - 1)$ is a factor of $f(x) = x^3 - 2x^2 - 5x + 6$, but $(x - 2)$ is not.

1. Evaluate $f(1)$:

$$f(1) = 1 - 2 - 5 + 6 = 0$$

Since $f(1) = 0$, $(x - 1)$ **is** a factor.

2. Evaluate $f(2)$:

$$f(2) = 8 - 8 - 10 + 6 = -4$$

Since $f(2) \neq 0$, $(x - 2)$ is **not** a factor (dividing by it would leave remainder -4).

3. Factorising a cubic fully

Now combine the two tools. The factor theorem finds you a way in; division does the rest.

THE METHOD

1. **Find one root** k by trying small values ($\pm 1, \pm 2, \dots$, and factors of the constant term) until $f(k) = 0$.
2. **Divide** $f(x)$ by $(x - k)$ — long division or grid — to get a quadratic.
3. **Factorise the quadratic** the usual way.

WORKED EXAMPLE 6.4 · FACTORISING A CUBIC

Factorise $f(x) = x^3 - 2x^2 - 5x + 6$ completely.

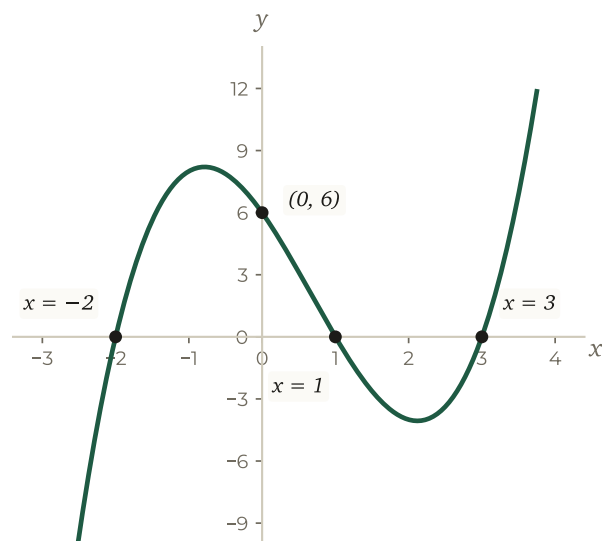
1. Try small values. $f(1) = 1 - 2 - 5 + 6 = 0$, so $(x - 1)$ is a factor.
2. Divide by $(x - 1)$ — by either method above:

$$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$$

3. Factorise the quadratic: $x^2 - x - 6 = (x - 3)(x + 2)$.

So the full factorisation is

$$f(x) = (x - 1)(x - 3)(x + 2)$$



$f(k) = 0$ means $(x - k)$ is a factor

The three factors give three roots — and three clean crossings.

COMMON MISTAKE

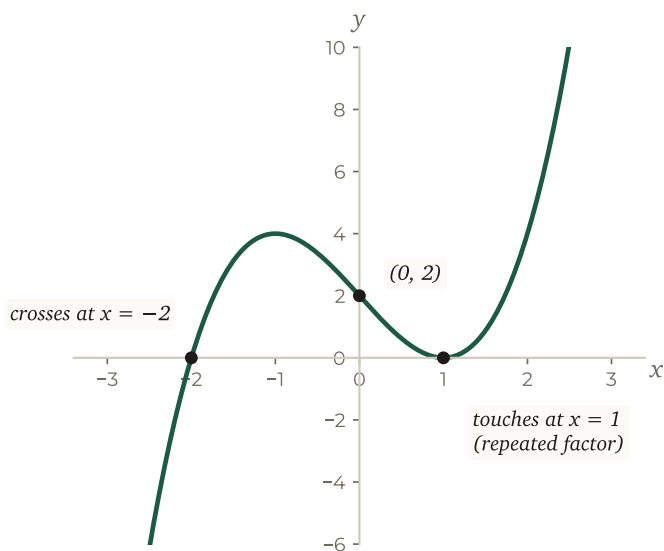
When hunting for the first root, only whole-number factors of the **constant term** can be integer roots. For $x^3 - 2x^2 - 5x + 6$, the constant is 6, so try $\pm 1, \pm 2, \pm 3, \pm 6$ — not, say, $x = 4$. It saves a lot of blind guessing.

4. Sketching the factorised cubic

Once factorised, the cubic sketches itself — the same crossing/touching rule from Using Graphs, now powered by the factor theorem.

KEY RESULT

- A **single** factor like $(x - 3)$ gives a clean **crossing** at $x = 3$.
- A **repeated** factor like $(x - 1)^2$ gives a **touch** — the curve meets the axis and turns back, as at a vertex.
- The y -intercept is $f(0)$; the highest power decides the far-left and far-right behaviour.



$(x + 2)(x - 1)^2$: a crossing at the single root, a touch at the repeated one.

WORKED EXAMPLE 6.5 · SKETCHING FROM FACTORS

Sketch $y = (x + 2)(x - 1)^2$, marking where it meets the axes.

1. Roots from the factors: $x = -2$ (single — a crossing) and $x = 1$ (repeated — a touch).
2. y -intercept at $x = 0$:

$$y = (2)(-1)^2 = 2$$

3. Expanding would begin with x^3 , so the curve rises from the bottom-left and climbs to the top-right.
4. Join them up: up through $(-2, 0)$, over a hump past $(0, 2)$, down to just touch the axis at $(1, 0)$, then away to the top right.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

Given $f(x) = x^3 + 2x^2 - 5x - 6$, evaluate:

(a) $f(2)$

(b) $f(-1)$

State which, if either, of $(x - 2)$ and $(x + 1)$ is a factor.

SOLUTION

(a) $f(2) = 8 + 8 - 10 - 6 = 0.$

(b) $f(-1) = -1 + 2 + 5 - 6 = 0.$

Both are zero, so by the factor theorem **both** $(x - 2)$ and $(x + 1)$ are factors.

Question 2

[3 marks]

Divide $x^2 + 5x + 6$ by $(x + 2)$.

SOLUTION

1. $x^2 \div x = x$. Multiply: $x(x + 2) = x^2 + 2x$. Subtract: $3x$.
2. Bring down $+6$. $3x \div x = 3$. Multiply: $3(x + 2) = 3x + 6$. Subtract: 0 .

The quotient is $x + 3$, remainder 0 : $x^2 + 5x + 6 = (x + 2)(x + 3)$.

Question 3

[2 marks]

The polynomial $g(x) = x^3 - 4x + 1$ is divided by $(x - 2)$. Use the remainder theorem to find the remainder.

SOLUTION

The remainder is $g(2)$:

$$g(2) = 8 - 8 + 1 = 1$$

Question 4

[2 marks]

A cubic is fully factorised as $y = (x - 4)(x + 1)(x - 2)$. Write down its three roots and its y -intercept.

SOLUTION

Roots where each factor is zero: $x = 4$, $x = -1$, $x = 2$.

y -intercept at $x = 0$: $y = (-4)(1)(-2) = 8$, so $(0, 8)$.

Standard**Question 5**

[4 marks]

Divide $2x^3 + 3x^2 - 11x - 6$ by $(x - 2)$.

SOLUTION

1. $2x^3 \div x = 2x^2$. Multiply: $2x^2(x - 2) = 2x^3 - 4x^2$. Subtract: $7x^2$.
2. Bring down $-11x$. $7x^2 \div x = 7x$. Multiply: $7x(x - 2) = 7x^2 - 14x$. Subtract: $3x$.
3. Bring down -6 . $3x \div x = 3$. Multiply: $3(x - 2) = 3x - 6$. Subtract: 0 .

Quotient $2x^2 + 7x + 3$, remainder 0:

$$2x^3 + 3x^2 - 11x - 6 = (x - 2)(2x^2 + 7x + 3)$$

Question 6

[5 marks]

Factorise $f(x) = x^3 - 3x^2 - 4x + 12$ completely.

SOLUTION

1. Try factors of 12. $f(2) = 8 - 12 - 8 + 12 = 0$, so $(x - 2)$ is a factor.
2. Divide: $x^3 - 3x^2 - 4x + 12 = (x - 2)(x^2 - x - 6)$.
3. Factorise the quadratic: $x^2 - x - 6 = (x - 3)(x + 2)$.

$$f(x) = (x - 2)(x - 3)(x + 2)$$

Question 7

[5 marks]

Factorise $g(x) = 2x^3 + 3x^2 - 11x - 6$ completely.

SOLUTION

1. From Question 5, $(x - 2)$ is a factor and $g(x) = (x - 2)(2x^2 + 7x + 3)$.
2. Factorise the quadratic: $2x^2 + 7x + 3 = (2x + 1)(x + 3)$.

$$g(x) = (x - 2)(2x + 1)(x + 3)$$

(Roots $x = 2, -\frac{1}{2}, -3$.)

Question 8

[4 marks]

Sketch the curve $y = (x - 2)^2(x + 1)$, marking the points where it meets the axes.

SOLUTION

1. Roots: $x = 2$ is repeated (a **touch**), $x = -1$ is single (a **crossing**).
2. y -intercept at $x = 0$: $y = (-2)^2(1) = 4$, so $(0, 4)$.
3. Positive x^3 leading term: rises from bottom-left to top-right.
4. The curve comes up through $(-1, 0)$, over to $(0, 4)$, down to touch the axis at $(2, 0)$, then back up.

Question 9

[4 marks]

The polynomial $f(x) = x^3 + ax^2 - x - 2$ has $(x - 1)$ as a factor. Find the value of a .

SOLUTION

By the factor theorem, $f(1) = 0$:

$$1 + a - 1 - 2 = 0 \implies a - 2 = 0 \implies a = 2$$

Exam style**Question 10**

[6 marks]

$$f(x) = 2x^3 - 5x^2 - x + 6$$

(a) Show that $(x + 1)$ is a factor of $f(x)$. [2]

(b) Hence factorise $f(x)$ completely. [4]

SOLUTION

(a) $f(-1) = 2(-1) - 5(1) - (-1) + 6 = -2 - 5 + 1 + 6 = 0$, so by the factor theorem $(x + 1)$ is a factor.
✓

(b) Divide $f(x)$ by $(x + 1)$:

$$2x^3 - 5x^2 - x + 6 = (x + 1)(2x^2 - 7x + 6)$$

Factorise the quadratic: $2x^2 - 7x + 6 = (2x - 3)(x - 2)$.

$$f(x) = (x + 1)(2x - 3)(x - 2)$$

Question 11

[5 marks]

The polynomial $p(x) = x^3 + bx^2 + cx - 10$ has factors $(x - 1)$ and $(x - 2)$. Find the values of b and c .

SOLUTION

Both factors mean $p(1) = 0$ and $p(2) = 0$.

$$1. \quad p(1) = 1 + b + c - 10 = 0 \implies b + c = 9.$$

$$2. \quad p(2) = 8 + 4b + 2c - 10 = 0 \implies 4b + 2c = 2 \implies 2b + c = 1.$$

3. Subtract the first from the second: $b = -8$, then $c = 9 - (-8) = 17$.

So $b = -8$, $c = 17$.

Question 12

[5 marks]

$$f(x) = x^3 - 6x^2 + 11x - 6$$

- (a) Factorise $f(x)$ completely. [3]
- (b) Hence solve $f(x) = 0$. [1]
- (c) Write down the coordinates of the point where $y = f(x)$ crosses the y -axis. [1]

SOLUTION

(a) Try factors of 6. $f(1) = 1 - 6 + 11 - 6 = 0$, so $(x-1)$ is a factor. Dividing: $f(x) = (x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3)$.

(b) $f(x) = 0$ at $x = 1, 2, 3$.

(c) $f(0) = -6$, so the curve crosses the y -axis at $(0, -6)$.