

YEAR 1 PURE · CHAPTER 10

Logarithms

Specification: Edexcel Pure 6 — Exponentials and logarithms · AQA section F

KEY FACTS — THIS CHAPTER

- A **logarithm** answers “what power?": $\log_a x = y$ means $a^y = x$ (with $a > 0$, $a \neq 1$, $x > 0$).
- So log is the **inverse** of raising to a power — the graph of $y = \log_a x$ is $y = a^x$ reflected in $y = x$.
- **Laws of logs:** $\log_a(xy) = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$, $\log_a(x^k) = k \log_a x$.
- Useful values: $\log_a a = 1$ and $\log_a 1 = 0$.
- To **solve** $a^x = b$, take logs of both sides: $x = \frac{\log b}{\log a}$.
- log usually means base 10; ln means base e (Chapter 11).

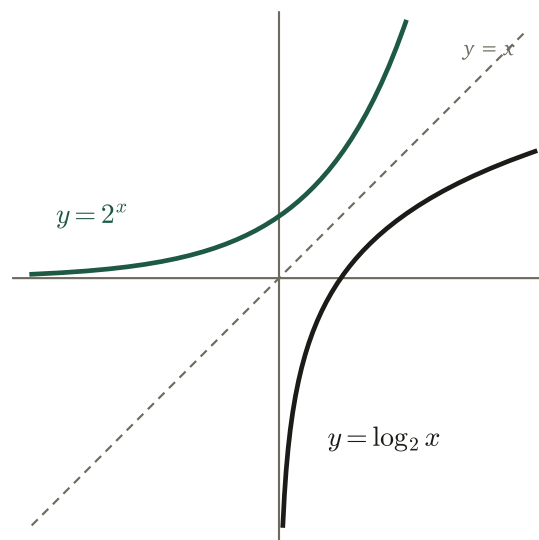
Indices answer “what do I get if I raise a to the power x ?” Logarithms run the question backwards: “what power of a gives me x ?” That single idea — log as the undo button for powers — is the whole chapter.

1. What a logarithm is

KEY RESULT

$$\log_a x = y \iff a^y = x$$

Read $\log_a x$ as “the power you raise a to, to get x ”. Because it undoes the exponential, $y = \log_a x$ is the **reflection** of $y = a^x$ in the line $y = x$.

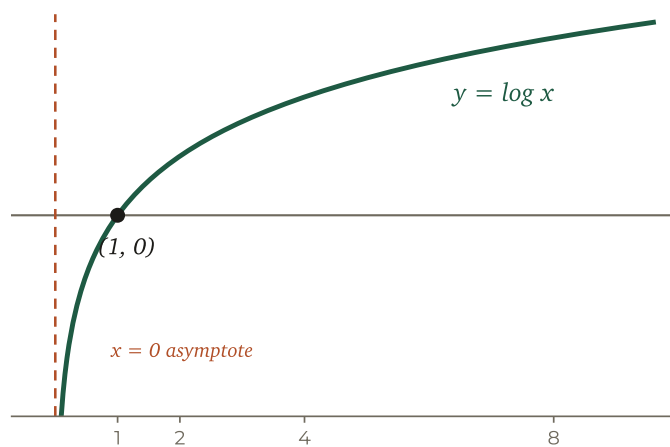


log is the inverse of the exponential — each graph is the other reflected in $y = x$.

WORKED EXAMPLE 10.1 · READING LOGS STRAIGHT OFF THE DEFINITION

Evaluate (a) $\log_2 16$, (b) $\log_{10} 1000$, (c) $\log_5 \frac{1}{25}$.

1. **(a)** “What power of 2 gives 16?” $2^4 = 16$, so $\log_2 16 = 4$.
2. **(b)** $10^3 = 1000$, so $\log_{10} 1000 = 3$.
3. **(c)** $5^{-2} = \frac{1}{25}$, so $\log_5 \frac{1}{25} = -2$.



Every log graph passes through (1, 0) and has the y-axis as an asymptote.

2. The laws of logarithms

The laws of logs are the index laws seen in a mirror — which is exactly why they look so similar.

KEY RESULT

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a(x^k) = k \log_a x$$

Plus the two you'll use constantly: $\log_a a = 1$ and $\log_a 1 = 0$.

WHY IT WORKS

Write $x = a^m$ and $y = a^n$, so $\log_a x = m$ and $\log_a y = n$. Then $xy = a^m \cdot a^n = a^{m+n}$ by the index laws, and taking the log gives $\log_a(xy) = m + n = \log_a x + \log_a y$. The product law for logs is just the *multiplication* law for indices, read backwards.

WORKED EXAMPLE 10.2 · COMBINING LOGS INTO ONE

Write $\log_2 40 - \log_2 5$ as a single logarithm, and hence evaluate it.

1. The subtraction is the quotient law:

$$\log_2 40 - \log_2 5 = \log_2 \frac{40}{5} = \log_2 8$$

2. And $\log_2 8 = 3$ since $2^3 = 8$.

WORKED EXAMPLE 10.3 · USING THE POWER LAW

Express $2\log_a 3 + \log_a 2$ as a single logarithm.

1. The coefficient 2 goes inside as a power:

$$2\log_a 3 = \log_a 3^2 = \log_a 9$$

2. Then add using the product law:

$$\log_a 9 + \log_a 2 = \log_a 18$$

COMMON MISTAKE

There is **no** law for $\log(x + y)$ — it does **not** equal $\log x + \log y$ (that's the law for $\log(xy)$). Logs only simplify products, quotients and powers, never sums or differences *inside* the log.

3. Solving exponential equations

When the unknown is stuck in an exponent — like $3^x = 20$ — take the log of both sides to bring it down.

KEY RESULT

To solve $a^x = b$, take logs and use the power law:

$$\log a^x = \log b \implies x \log a = \log b \implies x = \frac{\log b}{\log a}$$

Any base works (your calculator has log and ln) — just be consistent on both sides.

WORKED EXAMPLE 10.4 · AN EXPONENTIAL EQUATION

Solve $3^x = 20$, giving your answer to 3 significant figures.

1. Take logs of both sides and drop the power down:

$$x \log 3 = \log 20 \implies x = \frac{\log 20}{\log 3}$$

2. Evaluate:

$$x = 2.73 \text{ (3 s.f.)}$$

WORKED EXAMPLE 10.5 · THE EXPONENT IS AN EXPRESSION

Solve $2^{x+1} = 7$, giving your answer to 3 significant figures.

1. Take logs; the whole exponent $x + 1$ comes down:

$$(x + 1) \log 2 = \log 7 \implies x + 1 = \frac{\log 7}{\log 2}$$

2. So

$$x = \frac{\log 7}{\log 2} - 1 = 1.81 \text{ (3 s.f.)}$$

4. Solving equations with logs

If the unknown sits *inside* logs, combine them into one log first, then undo it with the definition.

WORKED EXAMPLE 10.6 · A LOGARITHMIC EQUATION

Solve $\log_3 x + \log_3(x - 2) = 1$.

1. Combine using the product law:

$$\log_3 (x(x - 2)) = 1$$

2. Undo the log with the definition ($\log_3 N = 1$ means $N = 3^1$):

$$x(x - 2) = 3 \implies x^2 - 2x - 3 = 0$$

3. Factorise: $(x - 3)(x + 1) = 0$, so $x = 3$ or $x = -1$.
4. Check the domain: logs need **positive** inputs, so $x - 2 > 0$ rules out $x = -1$ (and $x = 3$ works).
The solution is $x = 3$.

COMMON MISTAKE

Always check your answers against the domain at the end. A log is only defined for a positive argument, so any solution making x or $(x-2)$ zero or negative must be **discarded** — even though it solved the quadratic.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers. Give non-exact answers to 3 s.f.

Warm-up**Question 1**

[3 marks]

Evaluate:

(a) $\log_3 81$

(b) $\log_{10} 0.01$

(c) $\log_7 1$

SOLUTION

(a) $3^4 = 81$, so $\log_3 81 = 4$.

(b) $10^{-2} = 0.01$, so $\log_{10} 0.01 = -2$.

(c) $\log_a 1 = 0$ for any base, so $\log_7 1 = 0$.

Question 2

[3 marks]

Write as a single logarithm: $\log_5 6 + \log_5 4 - \log_5 3$.

SOLUTION

Add as a product, subtract as a quotient:

$$\log_5 6 + \log_5 4 - \log_5 3 = \log_5 \frac{6 \times 4}{3} = \log_5 8$$

Question 3

[2 marks]

Express $3 \log_a 2$ as a logarithm of a single number.

SOLUTION

The coefficient becomes a power: $3 \log_a 2 = \log_a 2^3 = \log_a 8$.

Question 4

[2 marks]

Solve $\log_2 x = 5$.

SOLUTION

By the definition, $x = 2^5 = 32$.

Standard**Question 5**

[3 marks]

Solve $5^x = 40$, giving your answer to 3 significant figures.

SOLUTION

Take logs of both sides:

$$x \log 5 = \log 40 \implies x = \frac{\log 40}{\log 5} = 2.29 \text{ (3 s.f.)}$$

Question 6

[4 marks]

Solve $4^{2x-1} = 30$, giving your answer to 3 significant figures.

SOLUTION

Take logs; the whole exponent comes down:

$$(2x - 1) \log 4 = \log 30 \implies 2x - 1 = \frac{\log 30}{\log 4} = 2.4534 \dots$$

$$2x = 3.4534 \dots \implies x = 1.73 \text{ (3 s.f.)}$$

Question 7

[4 marks]

Given that $\log_a x = 2$ and $\log_a y = 5$, find:

(a) $\log_a(x^2y)$

(b) $\log_a \frac{y}{x}$

SOLUTION

(a) $\log_a(x^2y) = 2\log_a x + \log_a y = 2(2) + 5 = 9.$

(b) $\log_a \frac{y}{x} = \log_a y - \log_a x = 5 - 2 = 3.$

Question 8

[4 marks]

Solve $\log_2(x+6) = 3 + \log_2 x$.

SOLUTION

1. Write $3 = 3\log_2 2 = \log_2 8$, then move logs to one side:

$$\log_2(x+6) - \log_2 x = \log_2 8 \implies \log_2 \frac{x+6}{x} = \log_2 8$$

2. Equate the arguments:

$$\frac{x+6}{x} = 8 \implies x+6 = 8x \implies x = \frac{6}{7}$$

Question 9

[5 marks]

Solve $\log_4 x + \log_4(x-6) = 2$.

SOLUTION

1. Combine: $\log_4(x(x-6)) = 2$.
2. Undo the log: $x(x-6) = 4^2 = 16$, so $x^2 - 6x - 16 = 0$.
3. Factorise: $(x-8)(x+2) = 0$, giving $x = 8$ or $x = -2$.
4. The argument must be positive, so discard $x = -2$. Thus $x = 8$.

Exam style**Question 10**

[5 marks]

Solve $3^{2x} - 10(3^x) + 9 = 0$.

SOLUTION

1. This is a hidden quadratic. Let $u = 3^x$ (so $3^{2x} = u^2$):

$$u^2 - 10u + 9 = 0$$

2. Factorise: $(u - 1)(u - 9) = 0$, so $u = 1$ or $u = 9$.

3. Convert back to x :

$$- 3^x = 1 \Rightarrow x = 0; - 3^x = 9 \Rightarrow x = 2.$$

So $x = 0$ or $x = 2$.

Question 11

[6 marks]

$$\log_2 y = 3 + 2 \log_2 x$$

- (a) Express y in terms of x , with no logarithms. [4]

- (b) Hence find y when $x = 4$. [2]

SOLUTION

(a)

1. Turn the right-hand side into a single log: $3 = \log_2 8$ and $2 \log_2 x = \log_2 x^2$, so

$$\log_2 y = \log_2 8 + \log_2 x^2 = \log_2 (8x^2)$$

2. Equate arguments: $y = 8x^2$.

(b) When $x = 4$: $y = 8(4)^2 = 8 \times 16 = 128$.

Question 12

[5 marks]

The number of bacteria N after t hours is modelled by $N = 200 \times 3^t$.

- (a) State the number of bacteria at $t = 0$. [1]
- (b) Find the smallest whole number of hours after which the number exceeds 50 000. [4]

SOLUTION

(a) At $t = 0$, $N = 200 \times 3^0 = 200$.

(b) Solve $200 \times 3^t = 50\,000$:

$$3^t = 250 \implies t = \frac{\log 250}{\log 3} = 5.02\dots$$

So N passes 50 000 during the 6th hour. At $t = 5$ it is still below ($200 \times 3^5 = 48\,600$), so the smallest whole number of hours after which it exceeds 50 000 is 6 ($200 \times 3^6 = 145\,800$).