

YEAR 1 PURE • CHAPTER 14

Integration

Specification: Edexcel Pure 8 — Integration • AQA section H

KEY FACTS — THIS CHAPTER

- **Integration is the reverse of differentiation** — it undoes the derivative.
- **Power rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (for $n \neq -1$) — raise the power by one, then divide by the new power.
- An **indefinite** integral always carries a **constant of integration** $+c$.
- A **definite** integral $\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$ — no $+c$ needed.
- For $f(x) \geq 0$, $\int_a^b f(x) dx$ is the **area** under the curve between $x = a$ and $x = b$.
- Given $\frac{dy}{dx}$ and one point, integrate and use the point to **find** c , recovering the curve.

Differentiation finds a gradient from a curve. Integration runs the other way: given the gradient, recover the curve — and, remarkably, the same operation measures the **area** underneath it.

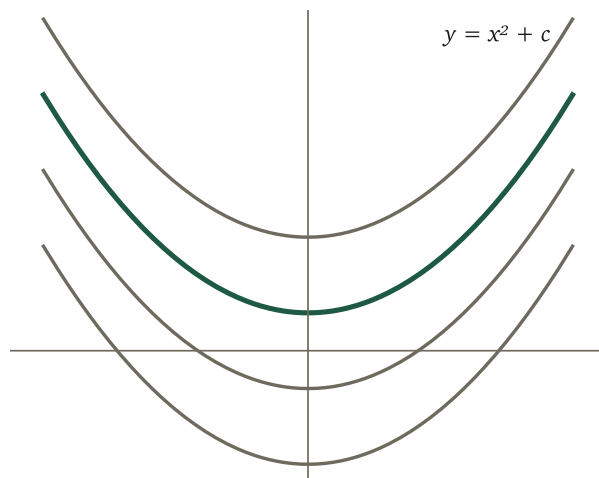
1. The reverse of differentiation

If differentiating x^3 gives $3x^2$, then integrating $3x^2$ should give back x^3 . There's a catch, though: any constant differentiates to zero, so we can never be sure it wasn't there — hence the $+c$.

KEY RESULT

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

Raise the power by one, then divide by the new power — the exact undo of the differentiation rule. The $+c$ is the **constant of integration**.



integrating $2x$ gives $x^2 + c$ — a whole family of curves

Integrating $2x$ gives $x^2 + c$ — a whole family of curves, one for each c .

WORKED EXAMPLE 14.1 • INTEGRATING A POLYNOMIAL

Find $\int (6x^2 + 2x - 5) dx$.

Integrate term by term, raising each power by one:

$$\int (6x^2 + 2x - 5) dx = \frac{6x^3}{3} + \frac{2x^2}{2} - 5x + c = 2x^3 + x^2 - 5x + c$$

WORKED EXAMPLE 14.2 • NEGATIVE AND FRACTIONAL POWERS

Find $\int \left(\frac{3}{x^2} + \sqrt{x} \right) dx$.

1. Rewrite in index form:

$$\int \left(3x^{-2} + x^{1/2} \right) dx$$

2. Apply the rule to each (power up by one, divide):

$$= \frac{3x^{-1}}{-1} + \frac{x^{3/2}}{3/2} + c = -3x^{-1} + \frac{2}{3}x^{3/2} + c$$

COMMON MISTAKE

Never drop the $+c$ on an indefinite integral — it's part of the answer, and exam mark schemes penalise its absence. (Definite integrals are the exception: the c cancels, so you leave it out there.)

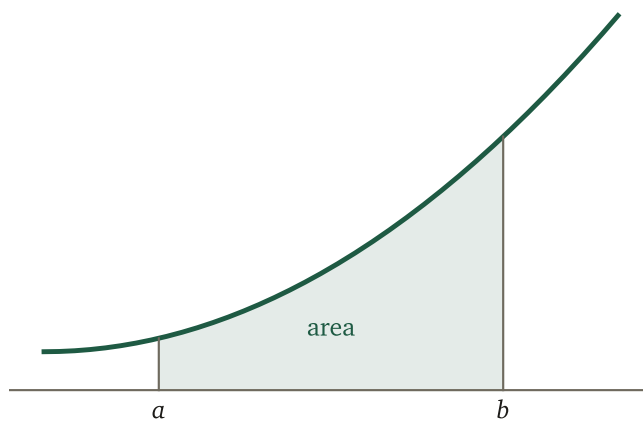
2. Definite integrals and area

Put limits on the integral and it becomes a **number** — and that number is the area under the curve.

KEY RESULT

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

where F is any integral of f . Integrate, substitute the top limit, subtract the bottom limit. For $f(x) \geq 0$ this equals the **area** between the curve and the x -axis from a to b .

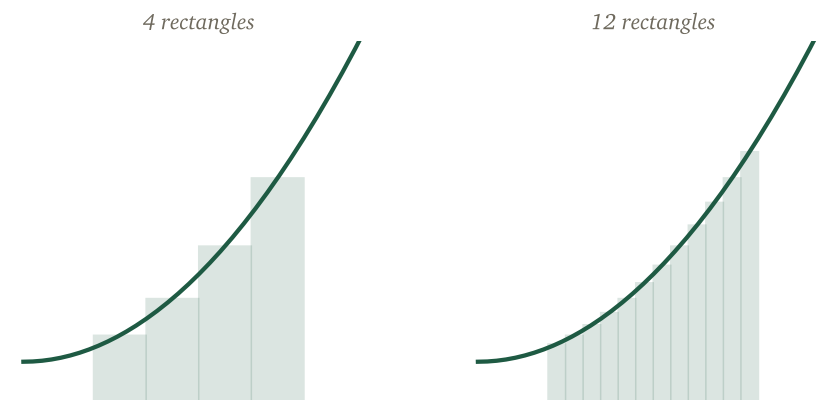


the shaded area is the definite integral from a to b

For a positive curve, the definite integral from a to b is the shaded area.

WHY IT WORKS

Slice the area into thin rectangles of width h and height $f(x)$. Their total is an approximation; as the rectangles get thinner the approximation tightens onto the true area — and that limit is exactly the integral. The same “shrink it to zero” move that built the derivative builds the area.



more, thinner rectangles get closer to the true area (the integral)

Thin rectangles approximate the area; in the limit they give the integral.

WORKED EXAMPLE 14.3 • EVALUATING A DEFINITE INTEGRAL

Evaluate $\int_1^3 3x^2 dx$.

1. Integrate (no $+c$ for a definite integral):

$$\int 3x^2 dx = x^3$$

2. Substitute the limits, top minus bottom:

$$[x^3]_1^3 = 3^3 - 1^3 = 27 - 1 = 26$$

WORKED EXAMPLE 14.4 • AREA UNDER A CURVE

Find the area under $y = x^2 + 1$ between $x = 0$ and $x = 2$.

Since $y > 0$ here, the area is the definite integral:

$$\int_0^2 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_0^2 = \left(\frac{8}{3} + 2 \right) - 0 = \frac{14}{3}$$

3. Finding a curve from its gradient

Because integration recovers a function from its derivative, a known point pins down the otherwise-mysterious $+c$.

KEY RESULT

Given $\frac{dy}{dx}$ and a point the curve passes through:

1. **Integrate** to get y (with $+c$);
2. **substitute** the known point to find c ;
3. write the full equation.

WORKED EXAMPLE 14.5 • RECOVERING THE CURVE

A curve passes through $(1, 3)$ and has gradient $\frac{dy}{dx} = 6x - 4$. Find its equation.

1. Integrate:

$$y = \int (6x - 4) dx = 3x^2 - 4x + c$$

2. Use $(1, 3)$ to find c :

$$3 = 3(1)^2 - 4(1) + c = -1 + c \implies c = 4$$

3. So the curve is $y = 3x^2 - 4x + 4$.

4. Area between a curve and a line

To find the area enclosed between two graphs, integrate the **difference** (top curve minus bottom curve) between their points of intersection.

KEY RESULT

The area between $y = f(x)$ (upper) and $y = g(x)$ (lower) from a to b is

$$\int_a^b [f(x) - g(x)] dx.$$

The a and b are usually where the graphs **meet** — find them by solving $f(x) = g(x)$ first.

WORKED EXAMPLE 14.6 • AREA BETWEEN A CURVE AND A LINE

Find the area enclosed between the curve $y = 4x - x^2$ and the x -axis.

1. Find where the curve meets the axis ($y = 0$):

$$4x - x^2 = x(4 - x) = 0 \implies x = 0 \text{ or } x = 4$$

2. Integrate between those limits:

$$\int_0^4 (4x - x^2) dx = \left[2x^2 - \frac{x^3}{3} \right]_0^4 = \left(32 - \frac{64}{3} \right) - 0 = \frac{32}{3}$$

COMMON MISTAKE

If part of the region dips **below** the x -axis, its integral comes out negative. Splitting the integral at the roots and taking the size of each piece avoids positive and negative areas cancelling — a classic trap in “total area” questions.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up**Question 1**

[3 marks]

Find:

(a) $\int x^3 dx$

(b) $\int 6x dx$

(c) $\int 4 dx$

SOLUTION

(a) $\frac{x^4}{4} + c.$

(b) $\frac{6x^2}{2} + c = 3x^2 + c.$

(c) $4x + c.$

Question 2

[3 marks]

Find $\int (4x^3 - 6x^2 + 2) dx.$ **SOLUTION**

$$\int (4x^3 - 6x^2 + 2) dx = x^4 - 2x^3 + 2x + c$$

Question 3

[3 marks]

Evaluate $\int_0^2 (3x^2 + 1) dx.$ **SOLUTION**

$$[x^3 + x]_0^2 = (8 + 2) - 0 = 10$$

Question 4

[2 marks]

Find $\int \frac{5}{x^2} dx$.

SOLUTION

Rewrite as $\int 5x^{-2} dx = \frac{5x^{-1}}{-1} + c = -\frac{5}{x} + c$.

Standard**Question 5**

[3 marks]

Find $\int \left(2\sqrt{x} + \frac{1}{x^3} \right) dx$.

SOLUTION

Rewrite as $\int (2x^{1/2} + x^{-3}) dx$:

$$= \frac{2x^{3/2}}{3/2} + \frac{x^{-2}}{-2} + c = \frac{4}{3}x^{3/2} - \frac{1}{2x^2} + c$$

Question 6

[4 marks]

Evaluate $\int_1^4 (6\sqrt{x}) dx$.

SOLUTION

1. $\int 6x^{1/2} dx = \frac{6x^{3/2}}{3/2} = 4x^{3/2}$.

2. Apply the limits:

$$\left[4x^{3/2} \right]_1^4 = 4(4^{3/2}) - 4(1) = 4(8) - 4 = 28$$

Question 7

[4 marks]

The area under $y = x^2$ between $x = 0$ and $x = k$ is 9. Find the value of k .

SOLUTION

$$\int_0^k x^2 dx = \left[\frac{x^3}{3} \right]_0^k = \frac{k^3}{3}$$

Set equal to 9: $\frac{k^3}{3} = 9 \Rightarrow k^3 = 27 \Rightarrow k = 3$.

Question 8

[4 marks]

A curve passes through (2, 5) and has $\frac{dy}{dx} = 3x^2 - 4x$. Find the equation of the curve.

SOLUTION

1. Integrate: $y = x^3 - 2x^2 + c$.
2. Use (2, 5): $5 = 8 - 8 + c \Rightarrow c = 5$.

So $y = x^3 - 2x^2 + 5$.

Question 9

[5 marks]

Find the area enclosed between the curve $y = 6x - x^2$ and the x -axis.

SOLUTION

1. Roots: $6x - x^2 = x(6 - x) = 0 \Rightarrow x = 0$ or $x = 6$.
2. Integrate between them:

$$\int_0^6 (6x - x^2) dx = \left[3x^2 - \frac{x^3}{3} \right]_0^6 = (108 - 72) - 0 = 36$$

Exam style**Question 10**

[6 marks]

The curve C has equation $y = x^3 - 4x$.

(a) Find the coordinates of the points where C meets the x -axis. [2]

(b) Evaluate $\int_0^2 (x^3 - 4x) dx$. [2]

(c) Explain why your answer to (b) is negative, and state the actual area between C and the x -axis from $x = 0$ to $x = 2$. [2]

SOLUTION

(a) $x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2) = 0$, so $x = 0, 2, -2$ — points $(0, 0)$, $(2, 0)$, $(-2, 0)$.

(b) $\left[\frac{x^4}{4} - 2x^2 \right]_0^2 = (4 - 8) - 0 = -4.$

(c) Between $x = 0$ and $x = 2$ the curve lies **below** the x -axis, so the integral is negative. The actual area is its size: 4.

Question 11

[6 marks]

The curve $y = x^2 + 2$ and the line $y = 3x$ intersect at two points.

- (a) Find the x -coordinates of the points of intersection. [2]
 (b) Find the area enclosed between the curve and the line. [4]

SOLUTION

(a) Set equal: $x^2 + 2 = 3x \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow (x - 1)(x - 2) = 0$, so $x = 1$ and $x = 2$.

(b) Between these, the line $3x$ is above the curve, so integrate (line – curve):

$$\begin{aligned} \int_1^2 [3x - (x^2 + 2)] dx &= \int_1^2 (-x^2 + 3x - 2) dx \\ &= \left[-\frac{x^3}{3} + \frac{3x^2}{2} - 2x \right]_1^2 = \left(-\frac{8}{3} + 6 - 4 \right) - \left(-\frac{1}{3} + \frac{3}{2} - 2 \right) = \frac{1}{6} \end{aligned}$$

Question 12

[5 marks]

Given that $\frac{dy}{dx} = \frac{12}{x^2} + 2$ and the curve passes through $(2, 7)$, find y in terms of x .

SOLUTION

1. Rewrite and integrate:

$$y = \int (12x^{-2} + 2) dx = \frac{12x^{-1}}{-1} + 2x + c = -\frac{12}{x} + 2x + c$$

2. Use $(2, 7)$: $7 = -\frac{12}{2} + 4 + c = -6 + 4 + c = -2 + c \Rightarrow c = 9.$

So $y = -\frac{12}{x} + 2x + 9.$