

YEAR 1 PURE • CHAPTER 11

Exponential Models

Specification: Edexcel Pure 6 — Exponentials and logarithms • AQA section F

KEY FACTS — THIS CHAPTER

- $e \approx 2.718$ is the special base where $y = e^x$ has gradient **equal to its own height** at every point.
- $\ln x$ means $\log_e x$ — the inverse of e^x , so $\ln(e^x) = x$ and $e^{\ln x} = x$.
- **Growth/decay model:** $y = Ae^{kt}$ starts at A (when $t = 0$); $k > 0$ grows, $k < 0$ decays.
- **Doubling time** is $\frac{\ln 2}{k}$; **half-life** is $\frac{\ln 2}{|k|}$.
- **Reduction to linear form:** for $y = ab^x$, taking logs gives $\log y = (\log b)x + \log a$ — a straight line when $\log y$ is plotted against x .

Lots of real things grow or shrink at a rate proportional to their current size — populations, savings, radioactive decay, cooling drinks. They're all modelled by the exponential function, and the natural base e is what makes the maths clean.

1. The number e and $y = e^x$

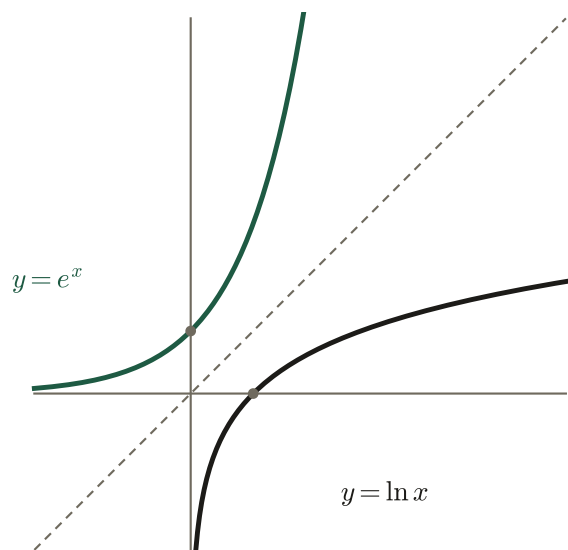
Among all the curves $y = a^x$, exactly one has the elegant property that its **gradient equals its height** everywhere. That base is $e \approx 2.718$.

KEY RESULT

For $y = e^x$, the gradient at any point equals the y -value there:

$$\frac{dy}{dx} = e^x = y$$

This is exactly why e shows up whenever a quantity changes at a rate proportional to its current amount.



$y = e^x$ and its inverse $y = \ln x$, mirror images in $y = x$.

2. The natural logarithm

The inverse of e^x is the **natural logarithm** $\ln x$ (that is, $\log_e x$). It obeys every log law from Chapter 10 — it just has base e .

KEY RESULT

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x$$

So to undo e , take $\ln$; to undo $\ln$, raise e to the power. This is the key to solving any equation with e or $\ln$ in it.

WORKED EXAMPLE 11.1 • SOLVING WITH E AND LN

Solve $e^{2x} = 7$, giving your answer to 3 significant figures.

1. Take $\ln$ of both sides — it cancels the e :

$$\ln(e^{2x}) = \ln 7 \implies 2x = \ln 7$$

2. So

$$x = \frac{\ln 7}{2} = 0.973 \text{ (3 s.f.)}$$

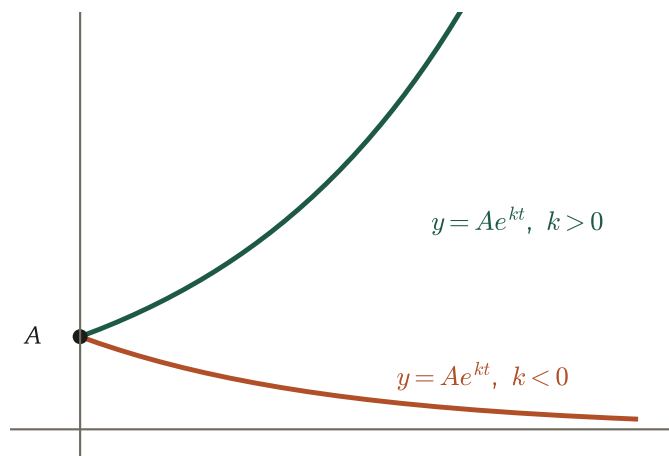
3. Exponential growth and decay

The workhorse model is $y = Ae^{kt}$. Reading off its constants in context is a core exam skill.

KEY RESULT

In $y = Ae^{kt}$:

- A is the **initial value** (at $t = 0$, since $e^0 = 1$);
- k is the **growth rate** — $k > 0$ grows, $k < 0$ decays;
- the **doubling time** (growth) is $\frac{\ln 2}{k}$, and the **half-life** (decay) is $\frac{\ln 2}{|k|}$.



growth ($k > 0$) rises; decay ($k < 0$) falls — both start at A

Same start A ; positive k grows, negative k decays.

WORKED EXAMPLE 11.2 • A POPULATION MODEL

A population is modelled by $P = 8000e^{0.05t}$, where t is in years. (a) Find the population after 12 years. (b) Find, to the nearest year, when the population reaches 20 000.

1. (a) Substitute $t = 12$:

$$P = 8000e^{0.05 \times 12} = 8000e^{0.6} = 14\,600 \text{ (3 s.f.)}$$

2. (b) Set $P = 20\,000$ and solve for t :

$$8000 e^{0.05t} = 20\,000 \implies e^{0.05t} = 2.5$$

3. Take $\ln$:

$$0.05t = \ln 2.5 \implies t = \frac{\ln 2.5}{0.05} = 18.3\dots$$

So the population reaches 20 000 after about 18 years.

WORKED EXAMPLE 11.3 · HALF-LIFE OF A DECAY

A radioactive sample has mass $m = 50 e^{-0.2t}$ grams, with t in days. Find its half-life.

1. The half-life is when the mass halves, from 50 to 25:

$$50 e^{-0.2t} = 25 \implies e^{-0.2t} = \frac{1}{2}$$

2. Take $\ln$:

$$-0.2t = \ln \frac{1}{2} = -\ln 2 \implies t = \frac{\ln 2}{0.2} = 3.47 \text{ days (3 s.f.)}$$

COMMON MISTAKE

The starting amount A is the value at $t = 0$, **not** the coefficient of t . In $P = 8000 e^{0.05t}$ the population starts at 8000 (because $e^0 = 1$), and grows at rate 0.05 — don't read 0.05 as a population.

4. Reduction to linear form

Raw exponential data looks like a curve, which is hard to fit by eye. Taking logs **straightens it into a line** — and a line you can find the equation of.

KEY RESULT

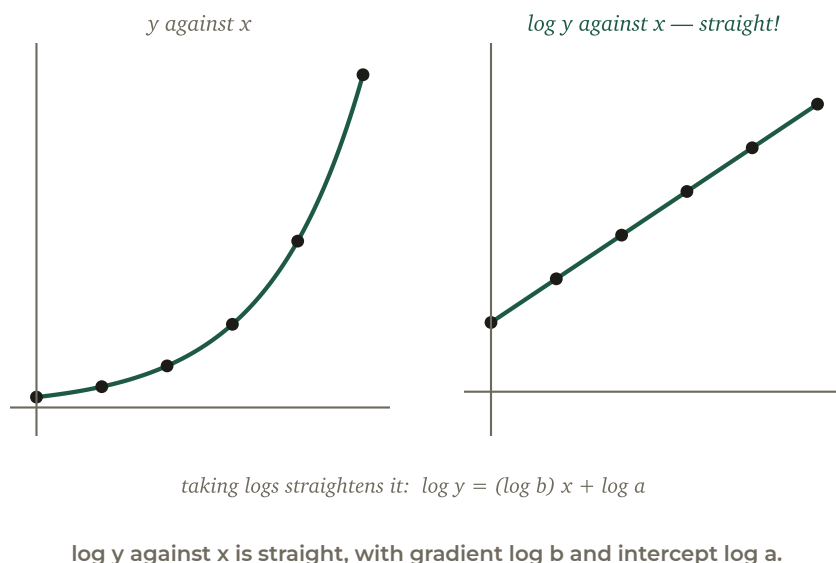
For a model $y = ab^x$, take logs of both sides:

$$\log y = \log a + x \log b$$

Plotting $\log y$ (vertical) against x (horizontal) gives a **straight line** with **gradient** $\log b$ and **vertical intercept** $\log a$.

WHY IT WORKS

The relationship $y = ab^x$ is curved, but applying log turns the product into a sum and brings the power down: $\log(ab^x) = \log a + \log(b^x) = \log a + x \log b$. That has the form $Y = mX + c$ with $Y = \log y$, so the transformed data is linear — and a ruler can fit a line where it couldn't fit a curve.

**WORKED EXAMPLE 11.4 • FINDING THE MODEL FROM A STRAIGHT LINE**

Data fits a model $y = ab^x$. When $\log_{10} y$ is plotted against x , the line has gradient 0.3 and vertical intercept 0.6. Find a and b .

1. Compare with $\log y = (\log b)x + \log a$:

- intercept: $\log_{10} a = 0.6 \implies a = 10^{0.6} = 3.98$ (3 s.f.) - gradient: $\log_{10} b = 0.3 \implies b = 10^{0.3} = 2.00$ (3 s.f.)

So the model is approximately $y = 3.98 \times 2.00^x$.

COMMON MISTAKE

The gradient and intercept give $\log a$ and $\log b$, **not** a and b themselves. You must raise the base to those values ($a = 10^{\text{intercept}}$, $b = 10^{\text{gradient}}$) to recover the constants.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers. Give non-exact answers to 3 s.f.

Warm-up**Question 1**

[3 marks]

Solve, giving exact answers:

(a) $e^x = 5$

(b) $\ln x = 3$

SOLUTION

(a) Take $\ln$: $x = \ln 5$.

(b) Raise e to both sides: $x = e^3$.

Question 2

[3 marks]

Solve $e^{3x} = 12$, giving your answer to 3 significant figures.

SOLUTION

$$3x = \ln 12 \implies x = \frac{\ln 12}{3} = 0.828 \text{ (3 s.f.)}$$

Question 3

[2 marks]

The value of a car is modelled by $V = 12000 e^{-0.15t}$ pounds, t years after purchase. State (a) the purchase value, (b) whether the model is growth or decay.

SOLUTION

- (a) At $t = 0$: $V = 12000 e^0 = \$12000$.
(b) $k = -0.15 < 0$, so the value **decays**.

Question 4

[2 marks]

Simplify (a) $\ln(e^4)$, (b) $e^{\ln 9}$.

SOLUTION

- (a) $\ln$ and e are inverses: $\ln(e^4) = 4$.
(b) Likewise $e^{\ln 9} = 9$.

Standard**Question 5**

[4 marks]

A population of bacteria is modelled by $N = 400 e^{0.08t}$, where t is in hours.

- (a) Find N when $t = 10$. [2]
(b) Find, to 3 s.f., the time taken for the population to reach 1000. [2]

SOLUTION

- (a) $N = 400 e^{0.8} = 890$ (3 s.f.).
(b) $400 e^{0.08t} = 1000 \Rightarrow e^{0.08t} = 2.5$, so

$$0.08t = \ln 2.5 \implies t = \frac{\ln 2.5}{0.08} = 11.5 \text{ (3 s.f.)}$$

Question 6

[4 marks]

A cup of coffee cools according to $T = 20 + 60 e^{-0.04t}$, where T is in $^{\circ}\text{C}$ and t in minutes.

- (a) State the initial temperature. [1]
(b) Find the temperature after 30 minutes. [1]
(c) Find, to the nearest minute, when the coffee reaches 50°C . [2]

SOLUTION

(a) At $t = 0$: $T = 20 + 60 = 80^\circ\text{C}$.

(b) $T = 20 + 60 e^{-1.2} = 38.1^\circ\text{C}$ (3 s.f.).

(c) $20 + 60 e^{-0.04t} = 50 \Rightarrow e^{-0.04t} = \frac{30}{60} = \frac{1}{2}$, so

$$-0.04t = \ln \frac{1}{2} = -\ln 2 \Rightarrow t = \frac{\ln 2}{0.04} = 17.3 \dots \approx 17 \text{ min}$$

Question 7

[4 marks]

A radioactive isotope decays as $m = m_0 e^{-0.05t}$ (t in years). Find its half-life, to 3 significant figures.

SOLUTION

Half-life is when $m = \frac{1}{2}m_0$:

$$e^{-0.05t} = \frac{1}{2} \Rightarrow -0.05t = -\ln 2 \Rightarrow t = \frac{\ln 2}{0.05} = 13.9 \text{ years}$$

Question 8

[4 marks]

Data fits the model $y = ab^x$. When $\log_{10} y$ is plotted against x , a straight line of gradient 0.5 and vertical intercept 1 is obtained. Find a and b .

SOLUTION

Comparing with $\log y = (\log b)x + \log a$:

- $\log_{10} a = 1 \Rightarrow a = 10^1 = 10$;
- $\log_{10} b = 0.5 \Rightarrow b = 10^{0.5} = 3.16$ (3 s.f.).

Exam style**Question 9**

[5 marks]

The mass of a culture is $M = 5 e^{0.6t}$ grams, t hours after the start.

- Find the doubling time, to 3 s.f. [3]
- Explain why the doubling time does not depend on the starting mass. [2]

SOLUTION

(a) Doubling means $M = 2 \times 5 = 10$:

$$5e^{0.6t} = 10 \Rightarrow e^{0.6t} = 2 \Rightarrow t = \frac{\ln 2}{0.6} = 1.16 \text{ hours}$$

(b) Setting $M = 2M_0$ gives $e^{kt} = 2$ whatever M_0 is — the M_0 cancels. So the doubling time $\frac{\ln 2}{k}$ depends only on the rate k , not the start.

Question 10

[6 marks]

The temperature of a liquid is modelled by $\theta = 75e^{-kt} + 18$, where t is in minutes. After 5 minutes the temperature is 60°C .

- (a) Find the value of k , to 3 s.f. [4]
 (b) Find the temperature after 15 minutes. [2]

SOLUTION

(a) At $t = 5$, $\theta = 60$:

$$75e^{-5k} + 18 = 60 \Rightarrow 75e^{-5k} = 42 \Rightarrow e^{-5k} = 0.56$$

Take $\ln$: $-5k = \ln 0.56$, so

$$k = -\frac{\ln 0.56}{5} = 0.116 \text{ (3 s.f.)}$$

(b) $\theta = 75e^{-0.116 \times 15} + 18 = 75e^{-1.74} + 18 = 31.2^\circ\text{C}$ (3 s.f.).

Question 11

[5 marks]

The table shows values fitting $y = ab^x$:

x	1	2	3
y	12	24	48

- (a) Explain how the table shows $b = 2$. [2]
 (b) Find a . [3]

SOLUTION

(a) Each time x increases by 1, y multiplies by 2 ($12 \rightarrow 24 \rightarrow 48$). For $y = ab^x$, a unit increase in x multiplies y by b , so $b = 2$.

(b) Use any data point, say $x = 1$, $y = 12$, with $b = 2$:

$$12 = a \times 2^1 \implies a = 6$$

So $y = 6 \times 2^x$.

Question 12

[6 marks]

A quantity is modelled by $y = 200 e^{kt}$. It is known that $y = 260$ when $t = 4$.

- (a) Find k , to 3 s.f. [3]
- (b) Find y when $t = 10$, to 3 s.f. [2]
- (c) Find, to 3 s.f., the value of t for which $y = 500$. [1]

SOLUTION

(a) $200 e^{4k} = 260 \Rightarrow e^{4k} = 1.3$, so

$$k = \frac{\ln 1.3}{4} = 0.0656 \text{ (3 s.f.)}$$

(b) $y = 200 e^{0.0656 \times 10} = 200 e^{0.656} = 385 \text{ (3 s.f.)}$.

(c) $200 e^{kt} = 500 \Rightarrow e^{kt} = 2.5$, so $t = \frac{\ln 2.5}{0.0656} = 14.0 \text{ (3 s.f.)}$.