

YEAR 1 PURE · CHAPTER 12

Differentiation

Specification: Edexcel Pure 7 — Differentiation · AQA section G

KEY FACTS — THIS CHAPTER

- The **derivative** $\frac{dy}{dx}$ is the **gradient of the curve** at each point — the slope of its tangent there.
- From **first principles**: $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.
- **Power rule**: if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$ — multiply by the power, then drop the power by one.
- Differentiate a polynomial **term by term**; constants differentiate to 0.
- For negative or fractional powers, **rewrite in index form first**, then apply the rule.
- The **second derivative** $\frac{d^2y}{dx^2}$ is the derivative of the derivative — the rate at which the gradient is changing.

Differentiation measures **how steep a curve is** at each point. A straight line has one gradient everywhere; a curve's gradient changes as you move along it, and the derivative is the function that gives it to you.

1. The gradient function

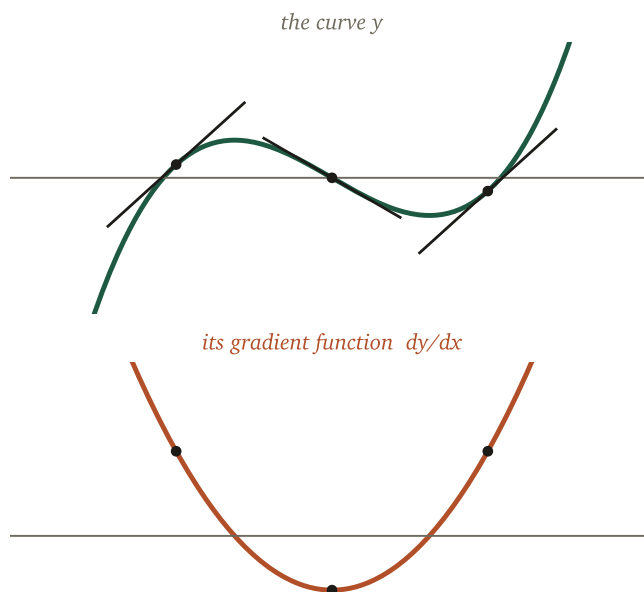
At each point of a curve, the **tangent** has a gradient. Collect those gradients for every x and you get a new function — the **gradient function** (or derivative), written $\frac{dy}{dx}$ or $f'(x)$.

KEY RESULT

$\frac{dy}{dx}$ is the gradient of the curve at each x . So:

- where the curve is **rising**, $\frac{dy}{dx} > 0$;
- where it is **falling**, $\frac{dy}{dx} < 0$;

- at a **turning point** (a flat tangent), $\frac{dy}{dx} = 0$.



Read the slope of the tangent at each point and you trace out the gradient function below.

This is the picture to keep in your head: the derivative is just the original curve's *slope*, plotted as its own graph.

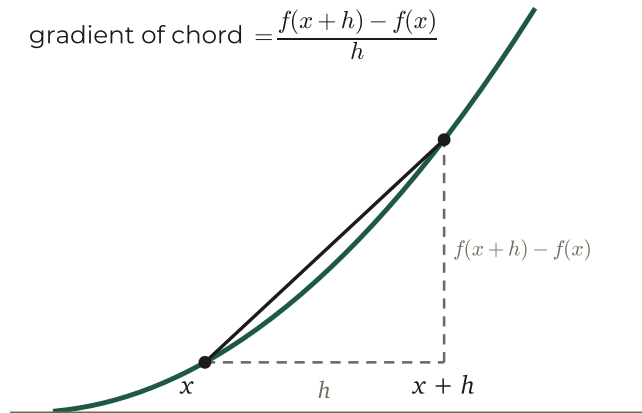
2. Differentiation from first principles

Before the shortcut, here's where the derivative comes from. Take two points on the curve a small distance h apart and find the gradient of the **chord** joining them; then shrink h to zero.

KEY RESULT

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The fraction is the gradient of the chord from x to $x+h$. As $h \rightarrow 0$ the chord swings round until it becomes the **tangent**, and its gradient becomes the derivative.



The chord's gradient is rise over run; let $h \rightarrow 0$ and it becomes the tangent's.

WORKED EXAMPLE 12.1 · DIFFERENTIATING x^2 FROM FIRST PRINCIPLES

Use first principles to differentiate $y = x^2$.

1. Form the chord gradient with $f(x) = x^2$:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

2. Expand and simplify the top:

$$= \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$$

3. Let $h \rightarrow 0$:

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} (2x + h) = 2x$$

3. The power rule

Doing that for every function would be exhausting — happily, a single rule handles all powers of x .

KEY RESULT

$$y = x^n \implies \frac{dy}{dx} = n x^{n-1}$$

Multiply by the power, then knock one off the power. For a sum, differentiate each term separately; a constant multiple stays, and a lone constant goes to 0 (its graph is flat).

WORKED EXAMPLE 12.2 · DIFFERENTIATING A POLYNOMIAL

Differentiate $y = 3x^4 - 2x^2 + 5x - 7$.

Apply the rule term by term:

$$\frac{dy}{dx} = 3(4x^3) - 2(2x) + 5(1) - 0 = 12x^3 - 4x + 5$$

WORKED EXAMPLE 12.3 · NEGATIVE AND FRACTIONAL POWERS

Differentiate $y = \frac{4}{x^2} + \sqrt{x}$.

1. Rewrite in index form first:

$$y = 4x^{-2} + x^{1/2}$$

2. Apply the power rule to each:

$$\frac{dy}{dx} = 4(-2)x^{-3} + \frac{1}{2}x^{-1/2} = -8x^{-3} + \frac{1}{2}x^{-1/2}$$

3. (Optional) write back in surd/fraction form:

$$\frac{dy}{dx} = -\frac{8}{x^3} + \frac{1}{2\sqrt{x}}$$

COMMON MISTAKE

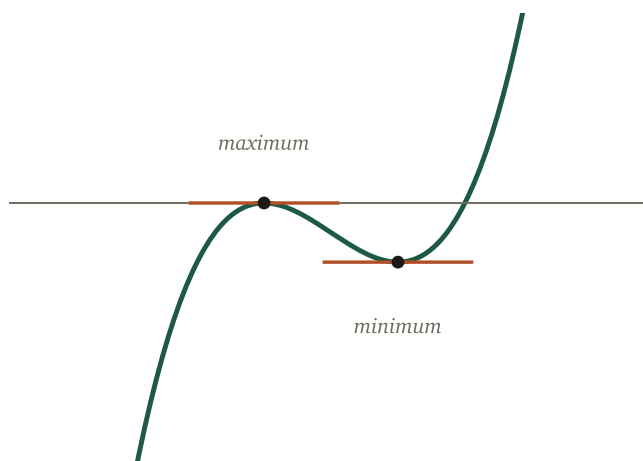
You **must** rewrite roots and fractions as powers before differentiating. $\sqrt{x}$ is $x^{1/2}$, and $\frac{1}{x^2}$ is x^{-2} — you can't differentiate them while they're still dressed as a root or a fraction.

4. Interpreting derivatives

Once you have $\frac{dy}{dx}$, substitute a value of x to get the **gradient at that point** — and differentiate again for the **second derivative**.

KEY RESULT

- $\frac{dy}{dx}$ evaluated at $x = a$ is the gradient of the curve at $x = a$.
- The **second derivative** $\frac{d^2y}{dx^2}$ (or $f''(x)$) is found by differentiating a second time. It measures how the gradient itself is changing.



at a stationary point the tangent is flat: $dy/dx = 0$

Where the gradient is zero, the tangent is flat — a turning point.

WORKED EXAMPLE 12.4 · GRADIENT AT A POINT, AND THE SECOND DERIVATIVE

For $y = x^3 - 4x$, find the gradient at $x = 2$ and the second derivative.

1. Differentiate:

$$\frac{dy}{dx} = 3x^2 - 4$$

2. The gradient at $x = 2$:

$$\left. \frac{dy}{dx} \right|_{x=2} = 3(2)^2 - 4 = 12 - 4 = 8$$

3. Differentiate again for the second derivative:

$$\frac{d^2y}{dx^2} = 6x$$

COMMON MISTAKE

A gradient is a *number* at a chosen point; the derivative is a *function*. “Find the gradient at $x = 2$ ” means differentiate **first**, then substitute $x = 2$ — never substitute into y before differentiating.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[3 marks]

Differentiate with respect to x :

(a) $y = x^5$

(b) $y = 3x^2$

(c) $y = 7$

SOLUTION

(a) $\frac{dy}{dx} = 5x^4.$

(b) $\frac{dy}{dx} = 6x.$

(c) A constant has zero gradient: $\frac{dy}{dx} = 0.$

Question 2

[3 marks]

Differentiate $y = 4x^3 - 5x^2 + 2x - 9$.

SOLUTION

Term by term:

$$\frac{dy}{dx} = 12x^2 - 10x + 2$$

Question 3

[2 marks]

Find the gradient of $y = x^2 - 3x$ at the point where $x = 4$.

SOLUTION

Differentiate, then substitute:

$$\frac{dy}{dx} = 2x - 3 \implies \left. \frac{dy}{dx} \right|_{x=4} = 8 - 3 = 5$$

Question 4

[2 marks]

Given $y = 2x^4$, find $\frac{d^2y}{dx^2}$.

SOLUTION

$$\frac{dy}{dx} = 8x^3, \quad \frac{d^2y}{dx^2} = 24x^2$$

Standard**Question 5**

[3 marks]

Differentiate $y = \frac{6}{x^3}$.

SOLUTION

Rewrite as $y = 6x^{-3}$, then apply the power rule:

$$\frac{dy}{dx} = 6(-3)x^{-4} = -18x^{-4} = -\frac{18}{x^4}$$

Question 6

[3 marks]

Differentiate $y = 4\sqrt{x}$.

SOLUTION

Rewrite as $y = 4x^{1/2}$:

$$\frac{dy}{dx} = 4 \times \frac{1}{2}x^{-1/2} = 2x^{-1/2} = \frac{2}{\sqrt{x}}$$

Question 7

[4 marks]

A curve has equation $y = x^3 - 6x^2 + 9x$. Find the values of x at which the gradient is zero.

SOLUTION

1. Differentiate:

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

2. Set equal to zero and solve:

$$3x^2 - 12x + 9 = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

So the gradient is zero at $x = 1$ and $x = 3$.

Question 8

[4 marks]

Differentiate $y = x^2(2x - 3)$.

SOLUTION

Expand first — the power rule needs separate powers of x :

$$y = 2x^3 - 3x^2 \implies \frac{dy}{dx} = 6x^2 - 6x$$

Question 9

[5 marks]

Use differentiation from first principles to find the derivative of $y = x^2 + 3x$.

SOLUTION

1. Form the chord gradient with $f(x) = x^2 + 3x$:

$$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 + 3(x+h)] - [x^2 + 3x]}{h}$$

2. Expand the top: $(x^2 + 2xh + h^2 + 3x + 3h) - (x^2 + 3x) = 2xh + h^2 + 3h$.

3. Divide by h and let $h \rightarrow 0$:

$$\frac{2xh + h^2 + 3h}{h} = 2x + h + 3 \xrightarrow{h \rightarrow 0} 2x + 3$$

So $\frac{dy}{dx} = 2x + 3$.

Exam style**Question 10**

[5 marks]

The curve C has equation $y = \frac{x^3 + 2x}{\sqrt{x}}$, for $x > 0$.

- (a) Show that y can be written as $x^{5/2} + 2x^{1/2}$. [2]

- (b) Hence find $\frac{dy}{dx}$. [3]

SOLUTION

- (a) Divide each term by $\sqrt{x} = x^{1/2}$:

$$y = \frac{x^3}{x^{1/2}} + \frac{2x}{x^{1/2}} = x^{3-1/2} + 2x^{1-1/2} = x^{5/2} + 2x^{1/2}$$

- (b) Apply the power rule:

$$\frac{dy}{dx} = \frac{5}{2}x^{3/2} + 2 \times \frac{1}{2}x^{-1/2} = \frac{5}{2}x^{3/2} + x^{-1/2}$$

Question 11

[5 marks]

The curve $y = x^3 - 3x^2 + 4$ passes through the point $P(1, 2)$.

- (a) Find the gradient of the curve at P . [3]
 (b) Find the value of x , other than at P , where the curve has the same gradient as at P . [2]

SOLUTION

(a) $\frac{dy}{dx} = 3x^2 - 6x$. At $x = 1$: $3 - 6 = -3$.

(b) Set the gradient equal to -3 :

$$3x^2 - 6x = -3 \implies 3x^2 - 6x + 3 = 0 \implies (x - 1)^2 = 0$$

The only solution is $x = 1$ (a repeated root), so there is **no other** point with gradient -3 — the curve has that gradient only at P .

Question 12

[6 marks]

For the curve $y = 2x^3 - 9x^2 + 12x$:

- (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]
 (b) Find the x -values where $\frac{dy}{dx} = 0$. [2]
 (c) Evaluate $\frac{d^2y}{dx^2}$ at each of those x -values. [1]

SOLUTION

(a)

$$\frac{dy}{dx} = 6x^2 - 18x + 12, \quad \frac{d^2y}{dx^2} = 12x - 18$$

(b) Solve $6x^2 - 18x + 12 = 0$:

$$x^2 - 3x + 2 = 0 \implies (x - 1)(x - 2) = 0 \implies x = 1 \text{ or } x = 2$$

(c) At $x = 1$: $\frac{d^2y}{dx^2} = 12 - 18 = -6$. At $x = 2$: $24 - 18 = 6$.

(The negative value flags a maximum at $x = 1$, the positive a minimum at $x = 2$ — a Chapter 13 idea.)