

YEAR 1 PURE • CHAPTER 5

Coordinate Geometry of Circles

Specification: Edexcel Pure 3 — Coordinate geometry • AQA section C

KEY FACTS — THIS CHAPTER

- A circle with centre (a, b) and radius r has equation $(x - a)^2 + (y - b)^2 = r^2$.
- To read off the centre and radius of $x^2 + y^2 + fx + gy + c = 0$, **complete the square**.
- Substituting a line into a circle gives a quadratic — its **discriminant** counts the intersections: two, one (a tangent) or none.
- A tangent is **perpendicular to the radius** at the point of contact.
- The **perpendicular bisector of any chord** passes through the centre.
- The **angle in a semicircle** is a right angle: if AB is a diameter and C lies on the circle, then $\angle ACB = 90^\circ$.

Circles turn everything you know about straight lines — gradients, perpendicularity, midpoints, distance — into one connected topic. Almost every circle question is secretly a straight-line question wearing a disguise.

1. The equation of a circle

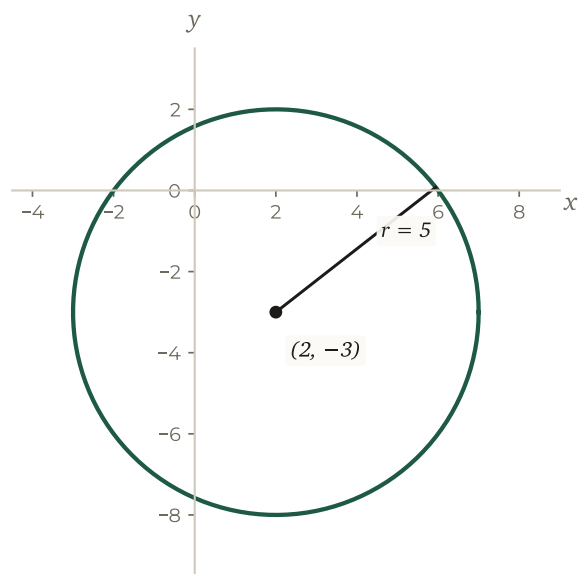
KEY RESULT

A circle with centre (a, b) and radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

WHY IT WORKS

A circle is the set of points at a fixed distance r from the centre. For any point (x, y) on the circle, the distance to (a, b) comes from Pythagoras: $\sqrt{(x - a)^2 + (y - b)^2} = r$. Squaring both sides gives the equation — so the circle equation is the distance formula, fixed at distance r .



The circle $(x - 2)^2 + (y + 3)^2 = 25$ — centre $(2, -3)$, radius 5.

Questions rarely hand you the tidy version, though. More often the equation arrives **expanded**, looking like $x^2 + y^2 + fx + gy + c = 0$ (the second form above), and your first job is to tidy it back up: complete the square in x and y separately, and the centre and radius fall out.

WORKED EXAMPLE 5.1 • FROM EXPANDED FORM TO CENTRE AND RADIUS

Find the centre and radius of the circle with equation $x^2 + y^2 - 4x + 6y - 12 = 0$.

1. Group the x -terms and y -terms, moving the constant across:

$$(x^2 - 4x) + (y^2 + 6y) = 12$$

2. Complete the square in each bracket:

$$x^2 - 4x = (x - 2)^2 - 4 \qquad y^2 + 6y = (y + 3)^2 - 9$$

3. Substitute back and tidy up:

$$(x - 2)^2 - 4 + (y + 3)^2 - 9 = 12 \implies (x - 2)^2 + (y + 3)^2 = 25$$

So the centre is $(2, -3)$ and the radius is $r = \sqrt{25} = 5$.

COMMON MISTAKE

Two classic slips. The centre takes the **opposite signs** to the brackets: $(x - 2)^2 + (y + 3)^2$ means centre $(2, -3)$, not $(-2, 3)$. And the right-hand side is r^2 , not r — if the equation ends $= 25$, the radius is 5.

2. Lines meeting circles

How many times does a line cross a circle? Twice, once, or not at all — and one method answers every version of the question. Substitute the line into the circle's equation and you're left with a quadratic, and quadratics you already know how to handle.

WORKED EXAMPLE 5.2 • INTERSECTION OF A LINE AND A CIRCLE

Find the points where the line $y = 2x - 3$ meets the circle $(x - 1)^2 + (y + 1)^2 = 20$.

1. Substitute $y = 2x - 3$ into the circle:

$$(x - 1)^2 + (2x - 3 + 1)^2 = 20$$

2. Notice $2x - 2 = 2(x - 1)$, so the second bracket is $4(x - 1)^2$:

$$(x - 1)^2 + 4(x - 1)^2 = 5(x - 1)^2 = 20$$

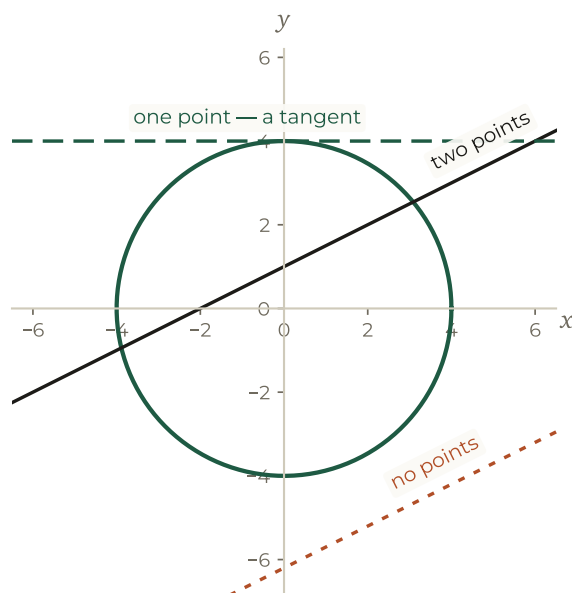
3. Solve: $(x - 1)^2 = 4$, so $x - 1 = \pm 2$, giving $x = 3$ or $x = -1$.
4. Substitute back into the **line** (always the easier equation) to find y : $x = 3$ gives $y = 3$; $x = -1$ gives $y = -5$.

The line meets the circle at $(3, 3)$ and $(-1, -5)$.

COUNTING INTERSECTIONS WITH THE DISCRIMINANT

Substituting a line into a circle gives a quadratic $px^2 + qx + s = 0$, and the discriminant $q^2 - 4ps$ counts the meeting points:

- $q^2 - 4ps > 0$ — the line **crosses** the circle twice
- $q^2 - 4ps = 0$ — the line **touches** the circle once: it is a **tangent**
- $q^2 - 4ps < 0$ — the line **misses** the circle



The three possibilities — two points, one point (a tangent), or none.

This is the standard route for “show that the line is a tangent” and “find the values of k for which the line meets the circle” questions: substitute, then work with the discriminant rather than solving fully.

3. Tangents and normals

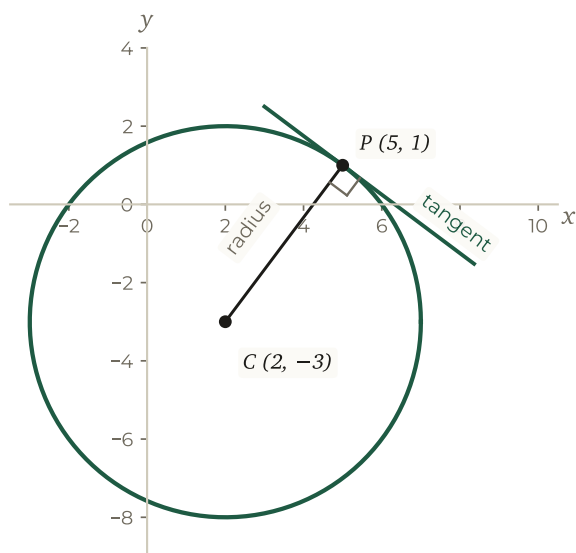
A tangent skims past a circle, touching it at exactly one point. You might expect finding its equation to need calculus — happily, circles hand you a shortcut for free.

KEY RESULT

A tangent to a circle is perpendicular to the radius at the point of contact. So at a point P on the circle:

$$m_{\text{tangent}} = -\frac{1}{m_{\text{radius}}}$$

The **normal** at P is the line along the radius — it passes through P and the centre.



The tangent at P is perpendicular to the radius CP.

This single fact does almost all the work in tangent questions: you never need calculus to find the tangent to a circle.

WORKED EXAMPLE 5.3 · FINDING A TANGENT TO A CIRCLE

Find the equation of the tangent to the circle $(x - 2)^2 + (y + 3)^2 = 25$ at the point $P(5, 1)$.

1. Check P lies on the circle: $(5 - 2)^2 + (1 + 3)^2 = 9 + 16 = 25$. ✓
2. Find the gradient of the radius from the centre $C(2, -3)$ to $P(5, 1)$:

$$m_{\text{radius}} = \frac{1 - (-3)}{5 - 2} = \frac{4}{3}$$

3. The tangent is perpendicular to the radius, so take the negative reciprocal:

$$m_{\text{tangent}} = -\frac{3}{4}$$

4. Build the line through $P(5, 1)$ with gradient $-\frac{3}{4}$ using $y - y_1 = m(x - x_1)$:

$$y - 1 = -\frac{3}{4}(x - 5)$$

Multiplying through by 4 and tidying: $3x + 4y = 19$.

COMMON MISTAKE

Don't differentiate, and don't use the gradient of the radius itself — the tangent's gradient is the **negative reciprocal** of the radius gradient. Step 1 is worth the five seconds: if the given point isn't actually on the circle, the question means something different (and your method must change).

4. Chords

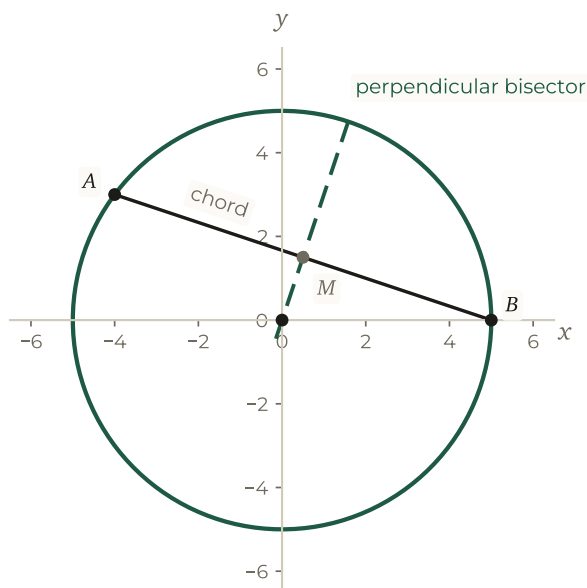
A **chord** is a straight line joining two points on a circle, cutting across the inside. One fact about chords does nearly all the heavy lifting in exam questions:

KEY RESULT

The perpendicular bisector of any chord passes through the centre of the circle. Equivalently: the perpendicular dropped from the centre to a chord cuts it exactly in half.

WHY IT WORKS

The centre is equidistant from every point on the circle — in particular from both ends of the chord. The set of all points equidistant from two points is the perpendicular bisector of the segment joining them, so the centre must sit on it.



The perpendicular bisector of the chord AB runs through the centre.

This is the key to “find the centre” questions: given a chord, build its perpendicular bisector and you have a line the centre is guaranteed to lie on. Two chords give two lines — and the centre is where

they cross.

WORKED EXAMPLE 5.4 · A CHORD'S PERPENDICULAR BISECTOR

The points $A(-4, 3)$ and $B(5, 0)$ lie on a circle. Show that the centre of the circle lies on the line $y = 3x$.

1. The centre lies on the perpendicular bisector of the chord AB , so find that line. Midpoint of AB :

$$M = \left(\frac{-4+5}{2}, \frac{3+0}{2} \right) = \left(\frac{1}{2}, \frac{3}{2} \right)$$

2. Gradient of AB :

$$m_{AB} = \frac{0 - 3}{5 - (-4)} = -\frac{1}{3},$$

so the perpendicular bisector has gradient 3.

3. The bisector is the line through M with gradient 3:

$$y - \frac{3}{2} = 3 \left(x - \frac{1}{2} \right) \implies y = 3x$$

The centre lies on the perpendicular bisector of AB , which is $y = 3x$. ■

5. Constructing circles from points

So far we've started from the circle's equation. Exam questions love running things the other way around: here are some points — now find the circle. Two set-ups cover nearly every version you'll meet.

From the ends of a diameter

If A and B are the ends of a diameter, the centre is the **midpoint** of AB and the radius is **half the distance** AB .

ANGLE IN A SEMICIRCLE

If AB is a diameter of a circle and C is any other point on the circle, then $\angle ACB = 90^\circ$. The converse also holds: if $\angle ACB = 90^\circ$, then C lies on the circle with diameter AB — a favourite “show that” step in exam questions.

WORKED EXAMPLE 5.5 · CIRCLE FROM A DIAMETER

The points $A(1, -2)$ and $B(7, 6)$ are the ends of a diameter of a circle. Find the equation of the circle.

1. The centre is the midpoint of AB :

$$\left(\frac{1+7}{2}, \frac{-2+6}{2}\right) = (4, 2)$$

2. The radius is the distance from the centre to either end:

$$r^2 = (7 - 4)^2 + (6 - 2)^2 = 9 + 16 = 25$$

So the circle is $(x - 4)^2 + (y - 2)^2 = 25$.

From two points and a radius

Given two points on a circle of known radius, the centre lies on the perpendicular bisector of the two points (it's equidistant from them) *and* at distance r from each. There are generally **two** circles that fit.

WORKED EXAMPLE 5.6 · TWO POINTS AND A RADIUS — TWO ANSWERS

A circle of radius $\sqrt{10}$ passes through $P(0, 2)$ and $Q(4, 4)$. Find the two possible equations of the circle.

1. The centre lies on the perpendicular bisector of PQ . Midpoint $(2, 3)$; gradient of PQ is $\frac{1}{2}$, so the bisector has gradient -2 :

$$y - 3 = -2(x - 2) \implies y = -2x + 7$$

2. Write the centre as $(a, 7 - 2a)$ and set its distance from $P(0, 2)$ equal to $\sqrt{10}$:

$$a^2 + (7 - 2a - 2)^2 = 10$$

3. Expand and simplify:

$$a^2 + (5 - 2a)^2 = 10$$

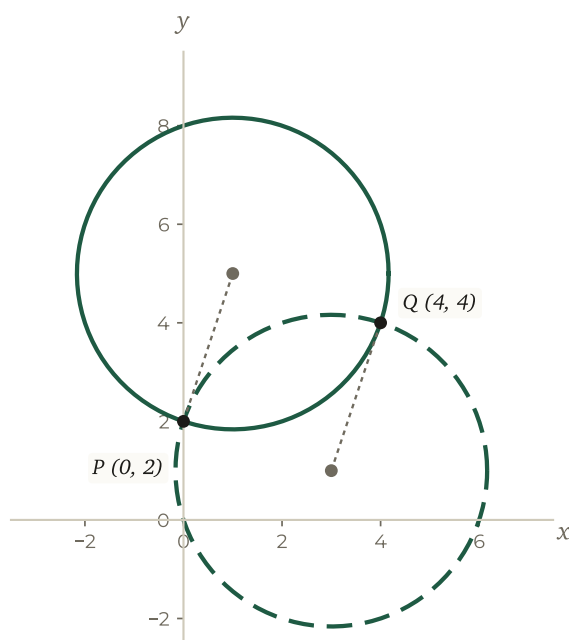
$$5a^2 - 20a + 15 = 0$$

$$a^2 - 4a + 3 = 0$$

4. Factorise: $(a - 1)(a - 3) = 0$, so $a = 1$ or $a = 3$, giving centres $(1, 5)$ and $(3, 1)$.

The two circles are

$$(x - 1)^2 + (y - 5)^2 = 10 \quad \text{and} \quad (x - 3)^2 + (y - 1)^2 = 10.$$



Both circles pass through P and Q with the same radius — expect two answers.

COMMON MISTAKE

If your working for this type of question produces only one centre, check it — a repeated root happens only when PQ is a diameter. Two distinct answers is the normal situation, and exam questions usually want **both**.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[2 marks]

Write down the centre and radius of the circle with equation $(x + 3)^2 + (y - 1)^2 = 49$.

SOLUTION

The equation matches $(x - a)^2 + (y - b)^2 = r^2$ with $a = -3$, $b = 1$, $r^2 = 49$.

Centre $(-3, 1)$, radius 7. Remember the signs flip: $(x + 3)^2 = (x - (-3))^2$.

Question 2

[2 marks]

Find the equation of the circle with centre $(4, -2)$ and radius $3\sqrt{2}$.

SOLUTION

Substitute into $(x - a)^2 + (y - b)^2 = r^2$, with $r^2 = (3\sqrt{2})^2 = 9 \times 2 = 18$:

$$(x - 4)^2 + (y + 2)^2 = 18$$

Question 3

[3 marks]

Find the centre and radius of the circle with equation $x^2 + y^2 - 6x + 2y - 6 = 0$.

SOLUTION

1. Group and move the constant: $(x^2 - 6x) + (y^2 + 2y) = 6$.
2. Complete the square: $(x - 3)^2 - 9 + (y + 1)^2 - 1 = 6$.
3. Tidy: $(x - 3)^2 + (y + 1)^2 = 16$.

Centre $(3, -1)$, radius 4.

Question 4

[2 marks]

Determine whether the point $(6, 2)$ lies inside, on, or outside the circle $(x - 3)^2 + (y + 1)^2 = 16$.

SOLUTION

Compare the squared distance from the point to the centre $(3, -1)$ with $r^2 = 16$:

$$(6 - 3)^2 + (2 - (-1))^2 = 9 + 9 = 18 > 16$$

The point is further from the centre than the radius, so $(6, 2)$ lies **outside** the circle. (No square roots needed — compare squared distances.)

Standard**Question 5**

[4 marks]

Find the coordinates of the points where the line $y = x + 2$ meets the circle $x^2 + y^2 = 10$.

SOLUTION

1. Substitute the line into the circle:

$$x^2 + (x + 2)^2 = 10$$

2. Expand and simplify:

$$2x^2 + 4x + 4 = 10 \implies x^2 + 2x - 3 = 0$$

3. Factorise: $(x + 3)(x - 1) = 0$, so $x = -3$ or $x = 1$.
4. Use the line for the y -values: $x = -3$ gives $y = -1$; $x = 1$ gives $y = 3$.

The points of intersection are $(-3, -1)$ and $(1, 3)$.

Question 6

[5 marks]

The circle C has equation $x^2 + y^2 - 8x - 4y + 15 = 0$.

- (a) Show that the point $P(2, 1)$ lies on C .
- (b) Find the equation of the tangent to C at P .

SOLUTION

(a) Substitute $(2, 1)$: $4 + 1 - 16 - 4 + 15 = 0$. ✓ So P lies on C .

(b)

1. Complete the square to find the centre: $(x - 4)^2 + (y - 2)^2 = 5$, so the centre is $(4, 2)$.
2. Gradient of the radius from $(4, 2)$ to $P(2, 1)$:

$$m_{\text{radius}} = \frac{1 - 2}{2 - 4} = \frac{1}{2}$$

3. Tangent gradient = negative reciprocal: $m = -2$.
4. Line through $P(2, 1)$ with gradient -2 :

$$y - 1 = -2(x - 2) \implies y = -2x + 5$$

Question 7

[5 marks]

The line $y = x + k$ is a tangent to the circle $x^2 + y^2 = 8$. Find the possible values of k .

SOLUTION

1. Substitute the line into the circle:

$$x^2 + (x + k)^2 = 8 \implies 2x^2 + 2kx + k^2 - 8 = 0$$

2. A tangent touches exactly once, so the discriminant is zero:

$$(2k)^2 - 4(2)(k^2 - 8) = 0$$

3. Simplify:

$$4k^2 - 8k^2 + 64 = 0 \implies k^2 = 16$$

So $k = 4$ or $k = -4$ — one tangent on each side of the circle.

Question 8

[4 marks]

The line $x = 6$ cuts the circle $(x - 3)^2 + (y - 2)^2 = 25$ at the points A and B .

- (a) Find the coordinates of A and B , and the length of the chord AB .

(b) Verify that the perpendicular from the centre to AB bisects the chord.

SOLUTION

(a)

1. Substitute $x = 6$: $(6 - 3)^2 + (y - 2)^2 = 25$, so $(y - 2)^2 = 16$.

2. $y - 2 = \pm 4$, giving $y = 6$ or $y = -2$.

So $A(6, 6)$ and $B(6, -2)$, and AB is vertical with length $6 - (-2) = 8$.

(b) The chord is vertical, so the perpendicular from the centre $(3, 2)$ is horizontal: the line $y = 2$. It meets AB at $(6, 2)$, which is exactly the midpoint of $A(6, 6)$ and $B(6, -2)$. ✓

Question 9

[6 marks]

A circle of radius 5 passes through the points $A(1, 2)$ and $B(5, 4)$. Find the two possible equations of the circle.

SOLUTION

1. The centre lies on the perpendicular bisector of AB . Midpoint $(3, 3)$; gradient of AB is $\frac{1}{2}$, so the bisector has gradient -2 :

$$y - 3 = -2(x - 3) \implies y = -2x + 9$$

2. Write the centre as $(a, 9 - 2a)$ and set its squared distance from $A(1, 2)$ equal to 25:

$$(a - 1)^2 + (7 - 2a)^2 = 25$$

3. Expand:

$$a^2 - 2a + 1 + 4a^2 - 28a + 49 = 25 \implies 5a^2 - 30a + 25 = 0$$

4. Divide by 5 and factorise: $a^2 - 6a + 5 = (a - 1)(a - 5) = 0$, so $a = 1$ or $a = 5$, giving centres $(1, 7)$ and $(5, -1)$.

The two circles are

$$(x - 1)^2 + (y - 7)^2 = 25 \quad \text{and} \quad (x - 5)^2 + (y + 1)^2 = 25.$$

Exam style**Question 10**

[5 marks]

The points $A(-2, 5)$ and $B(6, 1)$ are the ends of a diameter of the circle C .

- Find the equation of C . [3]
- Show that the point $D(0, 7)$ lies on C . [1]
- Write down the size of the angle ADB , giving a reason. [1]

SOLUTION**(a)**

- Centre = midpoint of $AB = \left(\frac{-2+6}{2}, \frac{5+1}{2}\right) = (2, 3)$.
- r^2 = squared distance from $(2, 3)$ to $B(6, 1)$: $(6-2)^2 + (1-3)^2 = 16 + 4 = 20$.
- Equation: $(x-2)^2 + (y-3)^2 = 20$.

(b) Substitute $D(0, 7)$: $(0-2)^2 + (7-3)^2 = 4 + 16 = 20$. ✓ So D lies on C .

(c) $\angle ADB = 90^\circ$ — the angle in a semicircle is a right angle, since AB is a diameter and D lies on the circle.

Question 11

[5 marks]

The line $y = mx$ is a tangent to the circle $(x-6)^2 + (y-3)^2 = 9$. Find the two possible values of m .

SOLUTION

- Substitute $y = mx$ into the circle:

$$(x-6)^2 + (mx-3)^2 = 9$$

- Expand and collect as a quadratic in x :

$$(1+m^2)x^2 - (12+6m)x + 36 = 0$$

- Tangency means the discriminant is zero:

$$(12+6m)^2 - 4(1+m^2)(36) = 0$$

4. Expand:

$$144 + 144m + 36m^2 - 144 - 144m^2 = 0$$

$$144m - 108m^2 = 0$$

5. Factorise: $36m(4 - 3m) = 0$, so $m = 0$ or $m = \frac{4}{3}$.

(Sense check: $m = 0$ is the x -axis, which sits exactly 3 below the centre $(6, 3)$ — a tangent, as expected.)

Question 12

[7 marks]

The circle C has equation $(x - 2)^2 + (y - 6)^2 = 40$, and the point $A(8, 4)$ lies on C .

- Find the equation of the tangent to C at A , giving your answer in the form $y = mx + c$. [4]
- The tangent meets the x -axis at P and the y -axis at Q . Find the area of the triangle OPQ , where O is the origin. [3]

SOLUTION

(a)

- Check A is on C : $(8 - 2)^2 + (4 - 6)^2 = 36 + 4 = 40$. ✓
- Gradient of the radius from the centre $(2, 6)$ to $A(8, 4)$:

$$m_{\text{radius}} = \frac{4 - 6}{8 - 2} = -\frac{1}{3}$$

- Tangent gradient = negative reciprocal: $m = 3$.
- Line through $A(8, 4)$: $y - 4 = 3(x - 8)$, so $y = 3x - 20$.

(b)

- On the x -axis, $y = 0$: $x = \frac{20}{3}$, so $P(\frac{20}{3}, 0)$.
- On the y -axis, $x = 0$: $y = -20$, so $Q(0, -20)$.
- The triangle has legs along the axes:

$$\text{Area} = \frac{1}{2} \times \frac{20}{3} \times 20 = \frac{200}{3} \approx 66.7$$