

YEAR 1 PURE • CHAPTER 7

Binomial Expansion

Specification: Edexcel Pure 4 — Sequences and series • AQA section D

KEY FACTS — THIS CHAPTER

- $(a + b)^n$ expands into $n + 1$ terms; the powers of a **fall** from n to 0 while the powers of b **rise** from 0 to n .
- The coefficients are row n of **Pascal's triangle**, where each entry is the sum of the two above it.
- Formally, the coefficient of the term in b^r is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, often written nC_r .
- **Binomial theorem:** $(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$.
- For $(1 + x)^n$ the terms are simply $1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots$
- Putting a small number in for x gives a fast, accurate **estimate** of a power — the more terms, the closer.

Multiplying out $(a + b)^2$ is easy; $(a + b)^7$ by hand is misery. The binomial expansion is the shortcut — a pattern that hands you every coefficient without multiplying a single bracket.

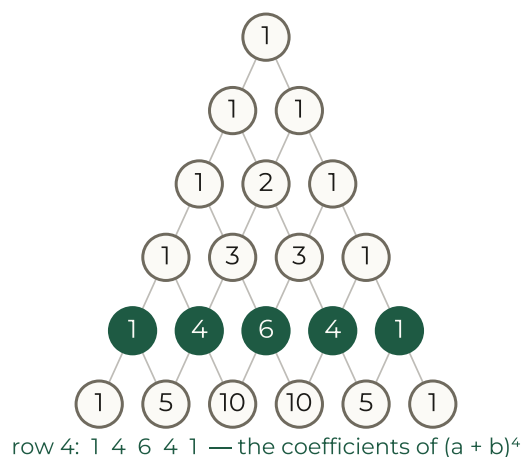
1. Pascal's triangle

Expand $(a + b)^n$ for $n = 0, 1, 2, 3$ and the coefficients line up into a famous triangle.

KEY RESULT

Each row of **Pascal's triangle** gives the coefficients of $(a + b)^n$. Every entry is the **sum of the two directly above it**, with 1s down each edge:

$$\begin{array}{c}
 1 \\
 1 \quad 1 \\
 1 \quad 2 \quad 1 \\
 1 \quad 3 \quad 3 \quad 1 \\
 1 \quad 4 \quad 6 \quad 4 \quad 1
 \end{array}$$



Each entry is the sum of the two above it; row 4 gives the coefficients of $(a + b)^4$.

Alongside the coefficients, the **powers** follow their own tidy rule: a counts down, b counts up, and every term's powers add to n .

powers of a fall $4 \rightarrow 0$, powers of b rise $0 \rightarrow 4$ (each term sums to 4)

a^4	$4a^3b$	$6a^2b^2$	$4ab^3$	b^4
a:4	a:3	a:2	a:1	a:0
b:0	b:1	b:2	b:3	b:4

In $(a + b)^4$: a 's power falls $4 \rightarrow 0$, b 's rises $0 \rightarrow 4$ — each term totalling 4.

WORKED EXAMPLE 7.1 • EXPANDING WITH PASCAL'S TRIANGLE

Expand $(a + b)^4$.

1. Row 4 of the triangle is 1, 4, 6, 4, 1.
2. Pair each coefficient with falling powers of a and rising powers of b :

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

WORKED EXAMPLE 7.2 • A BINOMIAL WITH A COEFFICIENT INSIDE

Expand $(2 + x)^3$.

1. Row 3 is 1, 3, 3, 1. Here $a = 2$ and $b = x$:

$$(2 + x)^3 = 1 \cdot 2^3 + 3 \cdot 2^2x + 3 \cdot 2x^2 + x^3$$

2. Work out each coefficient:

$$(2 + x)^3 = 8 + 12x + 6x^2 + x^3$$

COMMON MISTAKE

When a term like 2 or $3x$ sits inside the bracket, the power hits **all** of it: the second term of $(2 + x)^3$ is $3 \cdot 2^2 \cdot x = 12x$, not $3x$. Raise the whole thing, coefficient and all.

2. The binomial coefficient

Pascal's triangle is perfect for small n , but writing out twelve rows to expand $(a + b)^{12}$ is no fun. The formula nC_r jumps straight to any coefficient.

KEY RESULT

The coefficient of the term containing b^r in $(a + b)^n$ is

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$. It counts the ways of choosing which r of the n brackets contribute a b .

WHY IT WORKS

Expanding $(a + b)^n$ means multiplying n brackets and collecting like terms. A term with b^r comes from choosing b out of exactly r of the brackets (and a from the rest) — and the number of ways to make that choice is $\binom{n}{r}$. That count is the coefficient.

WORKED EXAMPLE 7.3 • A SINGLE TERM VIA THE FORMULA

Find the coefficient of x^3 in the expansion of $(1 + x)^7$.

1. For $(1 + x)^7$ the term in x^3 has coefficient $\binom{7}{3}$:

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

So the coefficient of x^3 is 35.

3. The binomial theorem

Putting it together gives the general formula — and for $(1 + x)^n$ it's especially clean.

KEY RESULT

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$$

In the common case $a = 1$, $b = x$:

$$(1 + x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n$$

WORKED EXAMPLE 7.4 • FIRST TERMS OF AN EXPANSION

Find the first four terms, in ascending powers of x , of $(1 + 2x)^6$.

1. Use $(1 + x)^n$ with the “ x ” being $2x$ and $n = 6$. The coefficients are $\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3} = 1, 6, 15, 20$.
2. Attach the powers of $2x$:

$$1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + \dots$$

3. Simplify each — remember the power hits the 2 as well:

$$1 + 12x + 60x^2 + 160x^3 + \dots$$

COMMON MISTAKE

$15(2x)^2$ is $15 \cdot 4x^2 = 60x^2$, not $30x^2$ — square the whole of $2x$. Forgetting to raise the inner coefficient is the classic binomial error, and it quietly wrecks every term after the first.

4. Estimating with the binomial expansion

A neat payoff: choosing a small value of x turns an expansion into a quick, surprisingly accurate numerical estimate.

THE IDEA

To estimate a power like $(1.02)^8$, write the base as $1 + x$ with x small, expand $(1 + x)^n$, and keep the first few terms. Because each term carries a higher power of a small number, the later terms are tiny — so a handful of terms is plenty.

WORKED EXAMPLE 7.5 • ESTIMATING A POWER

Use the first three terms of the expansion of $(1 + x)^8$ to estimate $(1.02)^8$.

1. The first three terms of $(1 + x)^8$ are

$$1 + 8x + \binom{8}{2}x^2 = 1 + 8x + 28x^2$$

2. Here $1.02 = 1 + 0.02$, so substitute $x = 0.02$:

$$1 + 8(0.02) + 28(0.02)^2 = 1 + 0.16 + 0.0112 = 1.1712$$

The true value is $1.171659\dots$, so three terms already give four-figure accuracy — the next term, in x^3 , contributes only about 0.0004.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

Warm-up

Question 1

[2 marks]

Write down row 5 of Pascal's triangle, and hence the coefficients of $(a + b)^5$.

SOLUTION

Each entry is the sum of the two above it. Row 5 is

$$1, 5, 10, 10, 5, 1$$

These are exactly the coefficients of $(a + b)^5$.

Question 2

[3 marks]

Expand $(a + b)^3$ and use it to expand $(x + 2)^3$.

SOLUTION

Row 3 is 1, 3, 3, 1, so $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

With $a = x$, $b = 2$:

$$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2)^2 + 2^3 = x^3 + 6x^2 + 12x + 8$$

Question 3

[3 marks]

Evaluate:

(a) $\binom{6}{2}$

(b) $\binom{6}{0}$

(c) $\binom{8}{8}$

SOLUTION

(a) $\binom{6}{2} = \frac{6 \times 5}{2 \times 1} = 15.$

(b) $\binom{6}{0} = 1$ (one way to choose nothing).

(c) $\binom{8}{8} = 1$ (one way to choose everything).

Question 4

[2 marks]

Write down the number of terms in the expansion of $(a + b)^{10}$, and the powers of a and b in its third term.

SOLUTION

$(a + b)^{10}$ has $10 + 1 = 11$ terms.

Counting from the first term a^{10} , the third term is the one in b^2 : it contains a^8b^2 (the powers always sum to 10).

Standard**Question 5**

[3 marks]

Expand $(2 + x)^4$ completely.

SOLUTION

Row 4 is 1, 4, 6, 4, 1, with $a = 2$, $b = x$:

$$2^4 + 4(2^3)x + 6(2^2)x^2 + 4(2)x^3 + x^4$$

$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

Question 6

[3 marks]

Find the coefficient of x^4 in the expansion of $(1 + x)^{10}$.

SOLUTION

The coefficient of x^4 is $\binom{10}{4}$:

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \frac{5040}{24} = 210$$

Question 7

[4 marks]

Find the first three terms, in ascending powers of x , of $(1 + 3x)^5$.

SOLUTION

Coefficients $\binom{5}{0}, \binom{5}{1}, \binom{5}{2} = 1, 5, 10$, attached to powers of $3x$:

$$1 + 5(3x) + 10(3x)^2 + \dots = 1 + 15x + 90x^2 + \dots$$

(Note $10(3x)^2 = 10 \times 9x^2 = 90x^2$.)

Question 8

[4 marks]

Find the coefficient of x^2 in the expansion of $(3 + 2x)^5$.

SOLUTION

The term in x^2 takes $b = 2x$ from 2 of the brackets and $a = 3$ from the other 3:

$$\binom{5}{2} 3^3 (2x)^2 = 10 \times 27 \times 4x^2 = 1080x^2$$

So the coefficient of x^2 is 1080.

Question 9

[3 marks]

Use the first three terms of the expansion of $(1 + x)^{10}$ to estimate $(1.01)^{10}$.

SOLUTION

First three terms: $1 + 10x + \binom{10}{2}x^2 = 1 + 10x + 45x^2$.

Substitute $x = 0.01$:

$$1 + 10(0.01) + 45(0.01)^2 = 1 + 0.1 + 0.0045 = 1.1045$$

(The true value is 1.10462...)

Exam style**Question 10**

[5 marks]

(a) Find the first four terms, in ascending powers of x , of $(1 - 2x)^7$. [3]

(b) Hence estimate $(0.98)^7$. [2]

SOLUTION

(a) Coefficients $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3} = 1, 7, 21, 35$, attached to powers of $(-2x)$:

$$1 + 7(-2x) + 21(-2x)^2 + 35(-2x)^3 + \dots$$

$$= 1 - 14x + 84x^2 - 280x^3 + \dots$$

(b) $0.98 = 1 - 2(0.01)$, so substitute $x = 0.01$:

$$\begin{aligned} &1 - 14(0.01) + 84(0.01)^2 - 280(0.01)^3 \\ &= 1 - 0.14 + 0.0084 - 0.00028 = 0.86812 \end{aligned}$$

(The true value is 0.868125...)

Question 11

[5 marks]

In the expansion of $(1 + kx)^6$, the coefficient of x^2 is 60. Given that $k > 0$, find the value of k .

SOLUTION

The coefficient of x^2 is $\binom{6}{2}k^2$:

$$\binom{6}{2}k^2 = 15k^2$$

Set equal to 60:

$$15k^2 = 60 \implies k^2 = 4 \implies k = 2 \quad (k > 0)$$

Question 12

[5 marks]

The first three terms of $(1 + ax)^n$, in ascending powers of x , are $1 + 12x + 60x^2$. Find the values of a and n .

SOLUTION

1. The x -term gives $\binom{n}{1}a = na = 12$.

2. The x^2 -term gives $\binom{n}{2}a^2 = \frac{n(n-1)}{2}a^2 = 60$.

3. From (1), $a = \frac{12}{n}$. Substitute into (2):

$$\frac{n(n-1)}{2} \cdot \frac{144}{n^2} = 60 \implies \frac{72(n-1)}{n} = 60$$

4. Solve: $72(n-1) = 60n \implies 72n - 72 = 60n \implies 12n = 72 \implies n = 6$.

5. Then $a = \frac{12}{6} = 2$.

So $a = 2$ and $n = 6$.