

## YEAR 1 PURE · CHAPTER 13

# Applications of Differentiation

Specification: Edexcel Pure 7 — Differentiation · AQA section G

## KEY FACTS — THIS CHAPTER

- The **tangent** at a point has gradient  $\frac{dy}{dx}$  there; the **normal** is perpendicular, with gradient  $-\frac{1}{dy/dx}$ .
- A **stationary point** is where  $\frac{dy}{dx} = 0$  — the curve is momentarily flat.
- **Classify** with the second derivative:  $\frac{d^2y}{dx^2} < 0$  is a **maximum**,  $\frac{d^2y}{dx^2} > 0$  is a **minimum**.
- A curve is **increasing** where  $\frac{dy}{dx} > 0$  and **decreasing** where  $\frac{dy}{dx} < 0$ .
- **Optimisation**: write the quantity as a function of one variable, differentiate, set it to 0, and solve — then check it's the max/min you want.

Chapter 12 gave you the derivative; this chapter spends it. Knowing the gradient at every point lets you draw tangents, find a curve's peaks and troughs, and solve the classic “what's the biggest/cheapest/fastest?” problems.

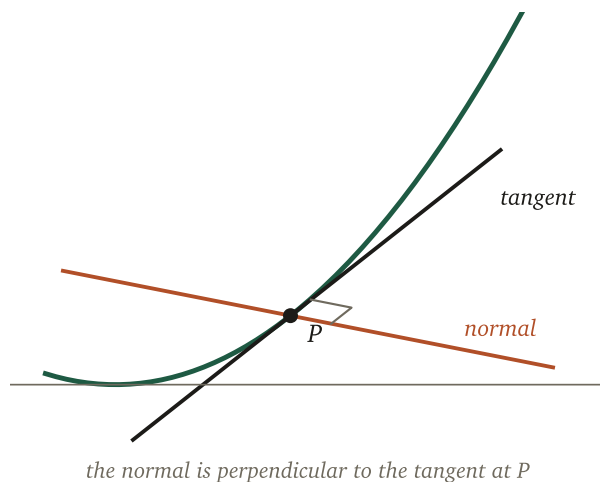
## 1. Tangents and normals

The derivative gives the **gradient of the tangent** at a point. From there it's just straight-line geometry (Chapter 4).

### KEY RESULT

At the point  $(x_1, y_1)$  on the curve:

- the **tangent** has gradient  $m = \frac{dy}{dx}$  there, and equation  $y - y_1 = m(x - x_1)$ ;
- the **normal** is perpendicular to it, with gradient  $-\frac{1}{m}$ .



The tangent follows the curve's gradient; the normal crosses it at a right angle.

#### WORKED EXAMPLE 13.1 · TANGENT AND NORMAL AT A POINT

Find the equations of the tangent and normal to  $y = x^2 - 3x + 2$  at the point where  $x = 3$ .

1. The point: at  $x = 3$ ,  $y = 9 - 9 + 2 = 2$ , so it's  $(3, 2)$ .
2. The gradient there:  $\frac{dy}{dx} = 2x - 3$ , so at  $x = 3$ ,  $m = 3$ .
3. **Tangent** (gradient 3 through  $(3, 2)$ ):

$$y - 2 = 3(x - 3) \implies y = 3x - 7$$

4. **Normal** (gradient  $-\frac{1}{3}$  through  $(3, 2)$ ):

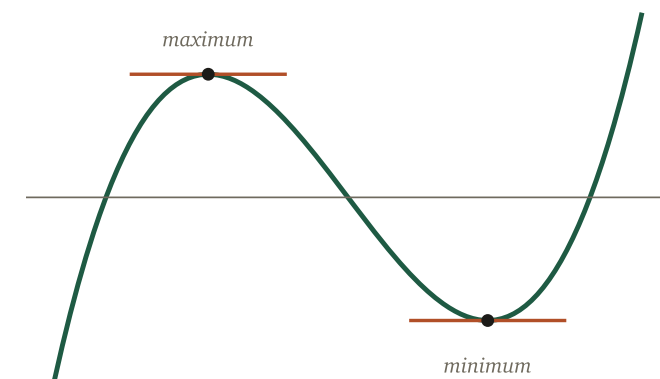
$$y - 2 = -\frac{1}{3}(x - 3) \implies y = -\frac{1}{3}x + 3$$

## 2. Stationary points

A **stationary point** is where the curve flattens out — a maximum, a minimum, or a point of inflection. All of them share one feature: the gradient is zero.

**KEY RESULT**

To find stationary points: solve  $\frac{dy}{dx} = 0$ . Each solution is the  $x$ -coordinate of a stationary point; substitute back into  $y$  for its height.



stationary points: where  $dy/dx = 0$ ; the 2nd derivative tells max from min

At a maximum and a minimum alike, the tangent is flat —  $dy/dx = 0$ .

**WORKED EXAMPLE 13.2 · FINDING THE STATIONARY POINTS**

Find the coordinates of the stationary points of  $y = x^3 - 3x$ .

1. Differentiate and set to zero:

$$\frac{dy}{dx} = 3x^2 - 3 = 0 \implies x^2 = 1 \implies x = \pm 1$$

2. Find the  $y$ -values:  $x = 1 \Rightarrow y = -2$ ;  $x = -1 \Rightarrow y = 2$ .

So the stationary points are  $(1, -2)$  and  $(-1, 2)$ .

**3. Maximum or minimum?**

Finding a stationary point is half the job; you also need to know **which kind**. The fastest test is the second derivative.

**KEY RESULT**

At a stationary point, look at  $\frac{d^2y}{dx^2}$ :

- $\frac{d^2y}{dx^2} < 0$  — a **maximum** (gradient is decreasing through it: the curve is bending down);
- $\frac{d^2y}{dx^2} > 0$  — a **minimum** (gradient increasing: bending up).

(If it's zero, the test is inconclusive — check the gradient either side instead.)

**WORKED EXAMPLE 13.3 · CLASSIFYING THE STATIONARY POINTS**

Classify the stationary points of  $y = x^3 - 3x$  from Example 13.2.

1. The second derivative:

$$\frac{d^2y}{dx^2} = 6x$$

2. Test each point:

- at  $x = -1$ :  $\frac{d^2y}{dx^2} = -6 < 0$ , so  $(-1, 2)$  is a **maximum**; - at  $x = 1$ :  $\frac{d^2y}{dx^2} = 6 > 0$ , so  $(1, -2)$  is a **minimum**.

**COMMON MISTAKE**

Substitute the stationary point's  $x$  into the **second** derivative, not the first (which is zero there by definition, so tells you nothing). And the sign is what matters — a negative second derivative means a maximum, which catches people out.

**4. Optimisation**

The big payoff: real problems asking for the largest area, smallest cost, maximum volume. The plan is always the same — get the quantity as a function of one variable, then differentiate to find the turning point.

**THE OPTIMISATION ROUTINE**

1. Write the quantity to optimise as a function of **one** variable (use any given constraint to eliminate the others).
2. Differentiate and set  $\frac{dy}{dx} = 0$ ; solve for the variable.
3. Check it's the max (or min) you want — the second derivative, or context.
4. Answer the question actually asked (often the optimal *value*, not just where).

**WORKED EXAMPLE 13.4 · THE LARGEST OPEN BOX**

An open box is made from a  $12\text{ cm} \times 12\text{ cm}$  square of card by cutting a square of side  $x$  from each corner and folding up the sides. Find the value of  $x$  that maximises the volume, and the maximum volume.

1. After cutting and folding, the base is  $(12 - 2x)$  by  $(12 - 2x)$  and the height is  $x$ , so

$$V = x(12 - 2x)^2$$

2. Expand and differentiate:

$$V = 144x - 48x^2 + 4x^3 \implies \frac{dV}{dx} = 144 - 96x + 12x^2$$

3. Set  $\frac{dV}{dx} = 0$ :

$$12x^2 - 96x + 144 = 0$$

$$x^2 - 8x + 12 = 0$$

$$(x - 2)(x - 6) = 0$$

So  $x = 2$  or  $x = 6$ . But  $x = 6$  leaves no base ( $12 - 2x = 0$ ), so  $x = 2$ .

4. Confirm it's a maximum:  $\frac{d^2V}{dx^2} = -96 + 24x$ , which at  $x = 2$  is  $-48 < 0$  — a maximum. ✓

The maximum volume is  $V = 2(12 - 4)^2 = 2 \times 64 = 128\text{ cm}^3$ .

**COMMON MISTAKE**

Don't stop at  $\frac{dV}{dx} = 0$ . Always discard solutions that don't make physical sense (here  $x = 6$  gives a flat box), confirm you've found a maximum, and read the question — it usually wants the maximum **volume**, not just the  $x$ .

**Practice**

Twelve questions in three tiers. Try each one properly before opening the solution — the solutions show full method, not just answers.

**Warm-up****Question 1**

[3 marks]

Find the equation of the tangent to  $y = x^2 + 1$  at the point  $(2, 5)$ .

**SOLUTION**

$\frac{dy}{dx} = 2x$ , so at  $x = 2$  the gradient is 4. Tangent through  $(2, 5)$ :

$$y - 5 = 4(x - 2) \implies y = 4x - 3$$

**Question 2**

[3 marks]

Find the  $x$ -coordinates of the stationary points of  $y = x^2 - 6x + 5$ .

**SOLUTION**

$\frac{dy}{dx} = 2x - 6 = 0 \implies x = 3$ . (One stationary point, at  $x = 3$ .)

**Question 3**

[3 marks]

A curve has  $\frac{dy}{dx} = 3x^2 - 12$ . State whether it is increasing or decreasing at (a)  $x = 1$ , (b)  $x = 3$ .

**SOLUTION**

(a) At  $x = 1$ :  $3 - 12 = -9 < 0$ , so **decreasing**.

(b) At  $x = 3$ :  $27 - 12 = 15 > 0$ , so **increasing**.

**Question 4**

[3 marks]

The curve  $y = x^2 - 4x$  has a stationary point. Find its coordinates and use the second derivative to state its nature.

**SOLUTION**

$$\frac{dy}{dx} = 2x - 4 = 0 \Rightarrow x = 2, y = 4 - 8 = -4. \text{ Point } (2, -4).$$

$$\frac{d^2y}{dx^2} = 2 > 0, \text{ so it is a \textbf{minimum}.}$$

**Standard****Question 5**

[4 marks]

Find the equation of the normal to  $y = x^2 - 5x + 6$  at the point where  $x = 1$ .

**SOLUTION**

1. At  $x = 1$ :  $y = 1 - 5 + 6 = 2$ , so the point is  $(1, 2)$ .
2.  $\frac{dy}{dx} = 2x - 5$ , so the tangent gradient at  $x = 1$  is  $-3$ ; the normal gradient is  $\frac{1}{3}$ .
3. Normal through  $(1, 2)$ :

$$y - 2 = \frac{1}{3}(x - 1) \implies y = \frac{1}{3}x + \frac{5}{3}$$

**Question 6**

[5 marks]

Find and classify the stationary points of  $y = 2x^3 - 3x^2 - 12x$ .

**SOLUTION**

1.  $\frac{dy}{dx} = 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0$ , so  $x = 2$  or  $x = -1$ .
2. Heights:  $x = 2 \Rightarrow y = 16 - 12 - 24 = -20$ ;  $x = -1 \Rightarrow y = -2 - 3 + 12 = 7$ .
3. Classify with  $\frac{d^2y}{dx^2} = 12x - 6$ :  
 - at  $x = -1$ :  $-18 < 0$ , so  $(-1, 7)$  is a **maximum**; - at  $x = 2$ :  $18 > 0$ , so  $(2, -20)$  is a **minimum**.

**Question 7**

[4 marks]

The curve  $y = x^3 - 12x + 3$  has a tangent that is parallel to the  $x$ -axis. Find the coordinates of the points where this happens.

**SOLUTION**

A tangent parallel to the  $x$ -axis has gradient 0:

$$\frac{dy}{dx} = 3x^2 - 12 = 0 \implies x = \pm 2$$

Heights:  $x = 2 \Rightarrow y = 8 - 24 + 3 = -13$ ;  $x = -2 \Rightarrow y = -8 + 24 + 3 = 19$ .

Points:  $(2, -13)$  and  $(-2, 19)$ .

**Question 8**

[5 marks]

A farmer has 40 m of fencing to enclose a rectangular pen against a wall (so only three sides are fenced). If the side perpendicular to the wall is  $x$  m, the area is  $A = x(40 - 2x)$ .

(a) Find the value of  $x$  that maximises the area. [3]

(b) Find the maximum area. [2]

**SOLUTION**

(a) Expand and differentiate:

$$A = 40x - 2x^2 \implies \frac{dA}{dx} = 40 - 4x = 0 \implies x = 10$$

$(\frac{d^2A}{dx^2} = -4 < 0, \text{ confirming a maximum.})$

(b)  $A = 10(40 - 20) = 10 \times 20 = 200 \text{ m}^2$ .

**Question 9**

[4 marks]

Show that the curve  $y = x^3 + 3x^2 + 5x - 2$  has no stationary points.

**SOLUTION**

$\frac{dy}{dx} = 3x^2 + 6x + 5$ . Stationary points need this to be zero, so check its discriminant:

$$\Delta = 6^2 - 4(3)(5) = 36 - 60 = -24 < 0$$

The quadratic has no real roots, so  $\frac{dy}{dx}$  is never zero — there are no stationary points. (Indeed  $\frac{dy}{dx} > 0$

everywhere, so the curve is always increasing.) ■

## Exam style

### Question 10

[6 marks]

The curve  $C$  has equation  $y = x^3 - 6x^2 + 9x + 1$ .

- (a) Find the coordinates of the two stationary points of  $C$ . [3]
- (b) Determine the nature of each. [2]
- (c) Find the equation of the tangent to  $C$  at  $x = 0$ . [1]

### SOLUTION

- (a)  $\frac{dy}{dx} = 3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0$ . At  $x = 1$ :  $y = 1 - 6 + 9 + 1 = 5$ .  
At  $x = 3$ :  $y = 27 - 54 + 27 + 1 = 1$ . Points:  $(1, 5)$  and  $(3, 1)$ .
- (b)  $\frac{d^2y}{dx^2} = 6x - 12$ : at  $x = 1$ ,  $-6 < 0$  (**maximum**); at  $x = 3$ ,  $6 > 0$  (**minimum**).
- (c) At  $x = 0$ :  $y = 1$  and the gradient is 9. Tangent:  $y = 9x + 1$ .

### Question 11

[6 marks]

A closed cylinder has radius  $r$  and a fixed volume. Its surface area is modelled by  $S = 2\pi r^2 + \frac{600}{r}$  (in  $\text{cm}^2$ ).

- (a) Find  $\frac{dS}{dr}$ . [2]
- (b) Find the value of  $r$  that minimises the surface area, to 3 s.f. [3]
- (c) Verify it is a minimum. [1]

### SOLUTION

- (a) Write  $S = 2\pi r^2 + 600r^{-1}$ :

$$\frac{dS}{dr} = 4\pi r - 600r^{-2} = 4\pi r - \frac{600}{r^2}$$

- (b) Set  $\frac{dS}{dr} = 0$ :

$$4\pi r = \frac{600}{r^2} \implies r^3 = \frac{600}{4\pi} = \frac{150}{\pi} \implies r = 3.63 \text{ (3 s.f.)}$$

(c)  $\frac{d^2S}{dr^2} = 4\pi + \frac{1200}{r^3} > 0$  for  $r > 0$ , so it is a minimum. ✓

**Question 12**

[6 marks]

A rectangle has its base on the  $x$ -axis and its top two corners on the curve  $y = 12 - x^2$  (with  $x > 0$  on the right). The corners are at  $(\pm x, 0)$  and  $(\pm x, 12 - x^2)$ , so its area is  $A = 2x(12 - x^2)$ .

(a) Find the value of  $x$  that maximises the area. [4]

(b) Find the maximum area. [2]

**SOLUTION**

(a) Expand and differentiate:

$$A = 24x - 2x^3 \implies \frac{dA}{dx} = 24 - 6x^2 = 0$$

$$x^2 = 4 \implies x = 2 \quad (x > 0)$$

(taking  $x > 0$ ;  $\frac{d^2A}{dx^2} = -12x < 0$  confirms a maximum).

(b)  $A = 2(2)(12 - 4) = 4 \times 8 = 32$ .