

YEAR 1 MECHANICS · CHAPTER 4

Variable Acceleration

Specification: Edexcel Paper 3 §7.4 — Calculus in kinematics · AQA section Q4

KEY FACTS — THIS CHAPTER

- When the acceleration **changes with time**, the *suvat* equations do not apply. Use calculus instead.
- **Differentiate** to go down: $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
- **Integrate** to go up: $v = \int a \, dt$ and $s = \int v \, dt$. Find each constant from a known value, such as “starts from rest” or “starts at O ”.
- $v = 0$ means the body is **momentarily at rest**. If v changes sign there, the body turns round, and its displacement is at a maximum or minimum.
- $\int_{t_1}^{t_2} v \, dt$ is the **displacement** between two times. For the **distance**, split the integral wherever $v = 0$ and add the sizes of the pieces.

In Chapter 2 the acceleration was constant: a straight v - t graph, and five equations to go with it. Real motion is rarely that tidy. A sprinter’s acceleration fades, and a car speeds up and brakes. When displacement, velocity or acceleration is given as a function of time, the tools are the two halves of calculus from Pure: differentiation (Chapter 12) and integration (Chapter 14).

1. Differentiating: from displacement to velocity to acceleration

Velocity is the rate of change of displacement, and acceleration is the rate of change of velocity. “Rate of change” means derivative.

KEY RESULT

$$v = \frac{ds}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

These are the gradients of the s - t and v - t graphs from Chapter 2, now for curves as well as straight lines. (Some books write r for displacement.)

WORKED EXAMPLE 4.1 · VELOCITY AND ACCELERATION FROM DISPLACEMENT

A particle moves along a straight line. Its displacement from a fixed point O at time t seconds is $s = t^3 - 6t^2 + 9t$ metres, for $t \geq 0$. Find the times when it is at rest, its displacement at those times, and its acceleration at those times.

Differentiate twice:

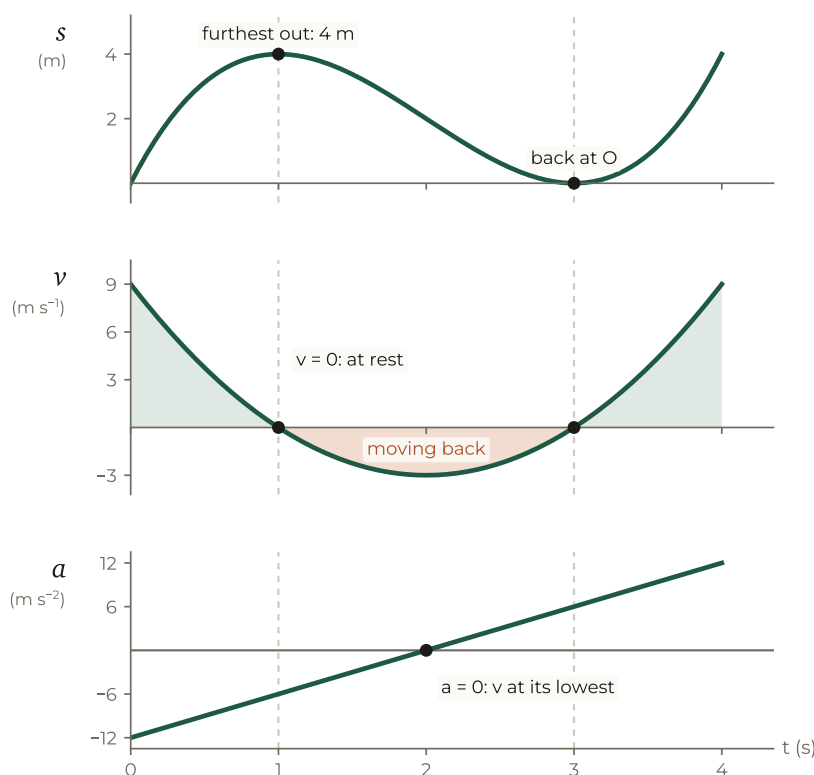
$$v = \frac{ds}{dt} = 3t^2 - 12t + 9 = 3(t-1)(t-3), \quad a = \frac{dv}{dt} = 6t - 12$$

At rest means $v = 0$, so $t = 1$ or $t = 3$. Then:

- at $t = 1$: $s = 1 - 6 + 9 = 4$ m and $a = -6$ m s⁻²;
- at $t = 3$: $s = 27 - 54 + 27 = 0$ m and $a = 6$ m s⁻².

IN CONTEXT

The particle stops 4 m from O after 1 s and turns back. It stops again after 3 s, exactly at O , and heads out once more.



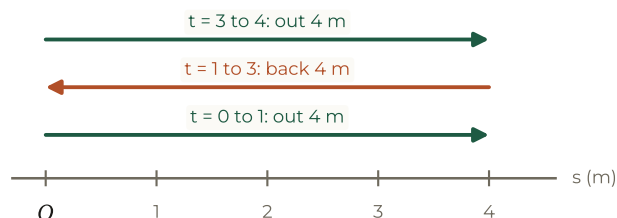
s , v and a on one time axis. Where $v = 0$ ($t = 1$ and 3), the s graph turns; where $a = 0$ ($t = 2$), v is lowest.

Read the three graphs from top to bottom and each one is the gradient of the one above it. That is differentiation, drawn.

2. Turning round: distance and displacement

KEY RESULT

- $v = 0$ where the body is **momentarily at rest**. If v changes sign there, the body turns round, and s has a maximum or minimum. This is the stationary-point idea from Chapter 13 of Pure.
- The **greatest speed** on an interval happens either where $a = 0$ or at an end of the interval.
- Displacement is where the body *is*. **Distance** is how far it has travelled, and every turn-round adds to it.



The same motion along the line: 4 m out, 4 m back, 4 m out again.

WORKED EXAMPLE 4.2 · DISTANCE TRAVELLED

For the particle in Example 4.1, find its displacement from O when $t = 4$, and the total distance it travels in the first 4 seconds.

$$s(4) = 64 - 96 + 36 = 4 \text{ m.}$$

The particle turns round at $t = 1$ and $t = 3$, so follow it stage by stage using $s(0) = 0$, $s(1) = 4$, $s(3) = 0$, $s(4) = 4$:

$$\text{distance} = |4 - 0| + |0 - 4| + |4 - 0| = 12 \text{ m}$$

IN CONTEXT

After 4 s the particle is 4 m from O , but it has travelled 12 m to get there.

COMMON MISTAKE

Never find the distance by working out s at the end time alone. Always check whether $v = 0$ somewhere *inside* the interval. If it is, the body turned round there, and you must work in stages.

3. Integrating: from acceleration back to displacement

Going the other way means integrating, and every integration brings a constant. The question always tells you enough to find it: “starts from rest” gives $v = 0$ at $t = 0$, and “starts at O ” gives $s = 0$ at $t = 0$.

KEY RESULT

$$v = \int a \, dt, \quad s = \int v \, dt$$

Add a constant each time and find it from the initial conditions. For the change in displacement between two times, use a definite integral:

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v \, dt$$

It is the signed area under the v - t graph, just as in Chapter 2.

WORKED EXAMPLE 4.3 · FROM ACCELERATION TO POSITION

A particle starts from rest at O . Its acceleration at time t seconds is $a = (6 - 2t) \, \text{m s}^{-2}$. Find its velocity and displacement at time t , its greatest velocity, and the time at which it returns to O , with its speed then.

$$v = \int (6 - 2t) \, dt = 6t - t^2 + c$$

At rest when $t = 0$, so $c = 0$ and $v = 6t - t^2$.

$$s = \int (6t - t^2) \, dt = 3t^2 - \frac{1}{3}t^3 + k$$

At O when $t = 0$, so $k = 0$.

Greatest velocity: where $a = 0$, i.e. $t = 3$, giving $v = 18 - 9 = 9 \text{ m s}^{-1}$.

Back at O : $s = t^2 \left(3 - \frac{1}{3}t\right) = 0$ gives $t = 9$. There, $v = 54 - 81 = -27$.

IN CONTEXT

The particle returns to O after 9 s, moving at 27 m s^{-1} back the way it came. The minus sign is the direction.

WHERE SUVAT COMES FROM

Integrate a **constant** acceleration a , starting with velocity u at O :

$$v = \int a \, dt = at + c, \quad v(0) = u \Rightarrow v = u + at$$

$$s = \int (u + at) \, dt = ut + \frac{1}{2}at^2 + k, \quad s(0) = 0 \Rightarrow s = ut + \frac{1}{2}at^2$$

The *suvat* equations are just the constant-acceleration case of this chapter. That is also why they fail when a varies.

COMMON MISTAKE

Don't use *suvat* when a depends on t . The give-away in a question is any expression in t , such as $a = 6 - 2t$ or $v = 3t^2 - 1$. If you see one, reach for calculus.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution. The solutions show the full method, not just answers. In every question a particle moves along a straight line, with t in seconds, displacements in metres and velocities in m s^{-1} .

Warm-up

Question 1

[3 marks]

The displacement of a particle from O is $s = 2t^3 - 5t^2 + 4$. Find its velocity and acceleration when $t = 2$.

SOLUTION

$$v = \frac{ds}{dt} = 6t^2 - 10t, \text{ so } v(2) = 24 - 20 = 4 \text{ m s}^{-1}.$$

$$a = \frac{dv}{dt} = 12t - 10, \text{ so } a(2) = 14 \text{ m s}^{-2}.$$

Question 2

[3 marks]

A particle has velocity $v = 3t^2 + 2t$. Find its acceleration when $t = 3$, and its displacement between $t = 0$ and $t = 2$.

SOLUTION

$$a = 6t + 2, \text{ so } a(3) = 20 \text{ m s}^{-2}.$$

$$\int_0^2 (3t^2 + 2t) dt = \left[t^3 + t^2 \right]_0^2 = 12 \text{ m. (The velocity is never negative on this interval, so this is also the distance.)}$$

Question 3

[3 marks]

A particle has acceleration $a = (4t - 1) \text{ m s}^{-2}$. When $t = 0$ its velocity is 2 m s^{-1} . Find an expression for its velocity, and the velocity when $t = 3$.

SOLUTION

$$v = \int (4t - 1) dt = 2t^2 - t + c. \text{ At } t = 0, v = 2, \text{ so } c = 2.$$

$$v = 2t^2 - t + 2, \text{ and } v(3) = 18 - 3 + 2 = 17 \text{ m s}^{-1}.$$

Question 4

[3 marks]

A particle has velocity $v = 12 - 3t^2$ for $t \geq 0$. Find when it is at rest, and its acceleration at that moment.

SOLUTION

$$12 - 3t^2 = 0 \Rightarrow t^2 = 4 \Rightarrow t = 2 \text{ (as } t \geq 0).$$

$$a = -6t, \text{ so } a(2) = -12 \text{ m s}^{-2}.$$

Standard**Question 5**

[6 marks]

A particle moves so that its displacement from O is $s = t^3 - 9t^2 + 24t$, for $t \geq 0$.

- Find the times at which the particle is at rest.
- Find its displacement from O at each of those times.

- (c) Find the total distance travelled in the first 5 seconds.

SOLUTION

(a) $v = 3t^2 - 18t + 24 = 3(t - 2)(t - 4)$, so $t = 2$ and $t = 4$.

(b) $s(2) = 8 - 36 + 48 = 20$ m and $s(4) = 64 - 144 + 96 = 16$ m.

(c) $s(0) = 0$ and $s(5) = 125 - 225 + 120 = 20$. It goes out 20 m, back 4 m, then out 4 m:

$$\text{distance} = 20 + 4 + 4 = 28 \text{ m}$$

Question 6

[6 marks]

A particle starts from rest at O with acceleration $a = (12 - 6t) \text{ m s}^{-2}$.

- (a) Find expressions for its velocity and displacement at time t .
 (b) Find its greatest velocity.
 (c) Find the time at which it returns to O .

SOLUTION

(a) $v = 12t - 3t^2 + c$, with $v = 0$ at $t = 0$, so $c = 0$. Then $s = 6t^2 - t^3 + k$, with $s = 0$ at $t = 0$, so $k = 0$.

(b) The greatest velocity is where $a = 0$: $t = 2$, so $v = 24 - 12 = 12 \text{ m s}^{-1}$.

(c) $s = t^2(6 - t) = 0$ gives $t = 6$ s.

Question 7

[5 marks]

A particle has velocity $v = t^2 - 4t + 3$. Find its displacement and the distance it travels between $t = 0$ and $t = 3$.

SOLUTION

$v = (t - 1)(t - 3)$, so the particle turns round at $t = 1$. Split the integral there.

$$\int_0^1 v \, dt = \left[\frac{1}{3}t^3 - 2t^2 + 3t \right]_0^1 = \frac{4}{3}, \quad \int_1^3 v \, dt = 0 - \frac{4}{3} = -\frac{4}{3}$$

Displacement $= \frac{4}{3} - \frac{4}{3} = 0$ m. Distance $= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$ m.

IN CONTEXT

The particle goes $\frac{4}{3}$ m out and comes straight back, ending where it started after travelling $\frac{8}{3}$ m.

Question 8

[5 marks]

A particle moves with **constant** acceleration a . At $t = 0$ it is at O with velocity u . Use integration to show that $v = u + at$ and $s = ut + \frac{1}{2}at^2$.

SOLUTION

$$v = \int a \, dt = at + c. \text{ When } t = 0, v = u, \text{ so } c = u \text{ and } v = u + at.$$

$$s = \int (u + at) \, dt = ut + \frac{1}{2}at^2 + k. \text{ When } t = 0, s = 0, \text{ so } k = 0 \text{ and } s = ut + \frac{1}{2}at^2. \blacksquare$$

Exam style**Question 9**

[9 marks]

A particle P leaves O at $t = 0$ and moves with velocity $v = 2t^2 - 14t + 20$.

- Find the acceleration of P when $t = 3$.
- Find the values of t at which P is at rest.
- Find the distance P travels between those two times.
- Find the total distance P travels in the first 6 seconds.

SOLUTION

(a) $a = 4t - 14$, so $a(3) = -2 \text{ m s}^{-2}$.

(b) $2(t^2 - 7t + 10) = 2(t - 2)(t - 5) = 0$, so $t = 2$ and $t = 5$.

(c) $s = \frac{2}{3}t^3 - 7t^2 + 20t$ (since $s = 0$ at $t = 0$). $s(2) = \frac{16}{3} - 28 + 40 = \frac{52}{3}$ and $s(5) = \frac{250}{3} - 175 + 100 = \frac{25}{3}$.

P does not turn round between these times, so the distance is $\frac{52}{3} - \frac{25}{3} = 9 \text{ m}$.

(d) $s(6) = 144 - 252 + 120 = 12$. The stages are out $\frac{52}{3}$, back 9, then out $12 - \frac{25}{3} = \frac{11}{3}$:

$$\frac{52}{3} + 9 + \frac{11}{3} = 30 \text{ m}$$

Question 10

[7 marks]

A car starts from rest. For $0 \leq t \leq 10$ its velocity is $v = 0.6t^2 - 0.04t^3$. After $t = 10$ it travels at a constant speed.

- (a) Show that the acceleration is zero when $t = 10$, and find the speed then.
- (b) Find the distance travelled in the first 10 s.
- (c) Find the total distance travelled in the first 15 s.

SOLUTION

(a) $a = 1.2t - 0.12t^2 = 0.12t(10 - t)$, which is 0 when $t = 10$. Then $v(10) = 60 - 40 = 20 \text{ m s}^{-1}$.

(b) $\int_0^{10} (0.6t^2 - 0.04t^3) dt = \left[0.2t^3 - 0.01t^4 \right]_0^{10} = 200 - 100 = 100 \text{ m}$.

(c) For the next 5 s it travels at 20 m s^{-1} , a further 100 m, so the total is 200 m.

IN CONTEXT

The car speeds up smoothly to 20 m s^{-1} over 100 m, then cruises. By $t = 15$ it has covered 200 m.

Question 11

[8 marks]

A particle passes O at $t = 0$ with velocity 5 m s^{-1} . Its acceleration is $a = (4 - 2t) \text{ m s}^{-2}$.

- (a) Find its velocity at time t .
- (b) Find the time at which it comes to instantaneous rest.
- (c) Find its greatest speed before that time.
- (d) Find its greatest displacement from O .

SOLUTION

(a) $v = 4t - t^2 + c$, with $v(0) = 5$, so $v = 5 + 4t - t^2$.

(b) $t^2 - 4t - 5 = 0 \Rightarrow (t - 5)(t + 1) = 0$, so $t = 5$.

(c) On $0 \leq t \leq 5$, $v \geq 0$, and v is greatest where $a = 0$: $t = 2$, giving $v = 5 + 8 - 4 = 9 \text{ m s}^{-1}$.

(d) $s = 5t + 2t^2 - \frac{1}{3}t^3$. The particle moves away from O until $t = 5$ and turns back after that, so the greatest displacement is $s(5) = 25 + 50 - \frac{125}{3} = \frac{100}{3} = 33.3 \text{ m}$ (3 s.f.).

Question 12

[9 marks]

Two particles P and Q leave O at the same time and move along the same straight line. Their displacements from O at time t are $s_P = t^2 + 2t$ and $s_Q = t^3 - 4t$.

- (a) Find the time at which they meet again, and how far from O they are then.
- (b) Find their speeds when they meet.
- (c) Show that the gap between them before they meet is greatest when their velocities are equal, and find that time.

SOLUTION

(a) $t^2 + 2t = t^3 - 4t \Rightarrow t^3 - t^2 - 6t = 0 \Rightarrow t(t-3)(t+2) = 0$, so $t = 3$. They meet $s = 9 + 6 = 15$ m from O .

(b) $v_P = 2t + 2 = 8 \text{ m s}^{-1}$ and $v_Q = 3t^2 - 4 = 23 \text{ m s}^{-1}$.

(c) The gap is $d = s_P - s_Q = -t^3 + t^2 + 6t$ for $0 < t < 3$. It is greatest where $\frac{dd}{dt} = 0$. But $\frac{dd}{dt} = v_P - v_Q$, so this happens exactly when the velocities are equal:

$$2t + 2 = 3t^2 - 4 \Rightarrow 3t^2 - 2t - 6 = 0$$

$$t = \frac{1 + \sqrt{19}}{3} = 1.79 \text{ s (3 s.f.)}$$

IN CONTEXT

P pulls ahead at first. The gap stops growing at $t = 1.79$ s, when Q has caught up to P 's speed. From then on Q is faster, and it closes the gap by $t = 3$.