

YEAR 1 MECHANICS • CHAPTER 1

Modelling in Mechanics

Specification: Edexcel Paper 3 §6.1 — Quantities and units in mechanics • AQA section P

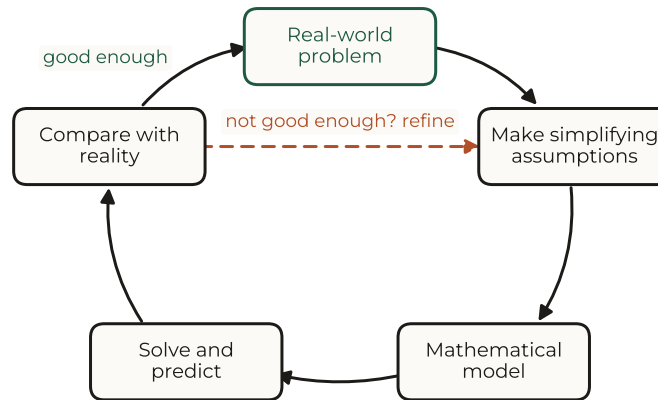
KEY FACTS — THIS CHAPTER

- A **mathematical model** is a simplified version of a real situation. You make **assumptions**, solve, compare with reality, and **refine** if you need to.
- The **SI base units** are the metre (m), the kilogram (kg) and the second (s). Every other unit is **derived** from them: velocity m s^{-1} , acceleration m s^{-2} , force $1 \text{ N} = 1 \text{ kg m s}^{-2}$.
- **Weight** is a force: $W = mg$ with $g = 9.8 \text{ m s}^{-2}$. Mass (kg) is not weight (N).
- **Particle**: all the mass sits at one point, so size, shape and rotation are ignored. **Light**: mass ignored. **Inextensible**: doesn't stretch. **Smooth**: no friction. **Rigid**: doesn't bend. **Uniform**: mass spread evenly.
- A **scalar** has size only (distance, speed, mass, time). A **vector** has size and direction (displacement, velocity, acceleration, force).
- For a velocity $a\mathbf{i} + b\mathbf{j}$, the **speed** is $\sqrt{a^2 + b^2}$, and its direction comes from $\tan \theta = \frac{b}{a}$.

Mechanics is the mathematics of things that move and the forces that move them. Real objects are messy: a football spins, deforms and drags air around it. Before any equation can say something useful, you have to decide what to ignore. That decision is called *modelling*, and this chapter is about making it well.

1. Mathematical models

A **model** keeps the features of a situation that matter most and deliberately throws away the rest. The throwing away is done by stating **assumptions**. A good model is simple enough to solve and accurate enough to be useful. No model is perfect.



The modelling cycle. If the prediction isn't good enough, go back and change the assumptions.

KEY RESULT

The modelling cycle

1. Start from the **real-world problem**.
2. Make **simplifying assumptions**.
3. Write the **mathematical model** (a diagram and equations).
4. **Solve** it and make a **prediction**.
5. **Compare** the prediction with reality. If it's good enough, use it. If not, **refine** the assumptions and go round again.

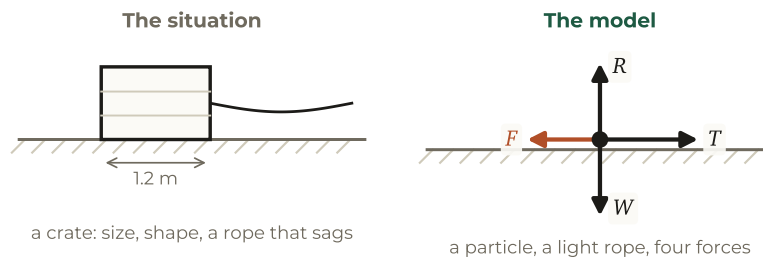
Exam questions test the last step as much as the others. Expect to be asked to *criticise* a model: say which assumption is least realistic and how it affects the answer. Then *refine* it: say what you would change.

2. Modelling assumptions

Mechanics has its own vocabulary of assumptions. Each word means something precise, and each one lets you leave something out of the maths.

Word	What it means	What it lets you ignore
Particle	the object's mass is concentrated at a single point	size, shape and rotation; air resistance too, usually

Word	What it means	What it lets you ignore
Light (string, rod, pulley)	its mass is negligible	the weight of the string; tension is the same all the way along it
Inextensible (string)	it does not stretch	changes in length; objects joined by it move with the same speed and acceleration
Smooth (surface, pulley)	there is no friction	friction forces
Rough (surface)	there is friction	nothing: friction must be included
Rigid (rod, body)	it does not bend or deform	changes of shape
Uniform (rod, body)	its mass is evenly spread	uneven mass: the weight acts at the centre
Bead	a particle threaded on a wire or string	its size, and the wire's thickness
Wire	a rigid, thin, light path an object moves along	its thickness
Peg	a fixed support a string passes over	its size; if smooth, friction too



Modelling a crate as a particle on a light rope. Four forces are left, all acting at one point.

MODELLING ASSUMPTION

Treating the crate as a particle means ignoring its size, so it can't tip or rotate, and every force acts at one point. **Treating the rope as light** means the rope has no weight, doesn't sag, and pulls equally hard all along its length.

COMMON MISTAKE

An assumption is not a fact, so don't write "the crate *is* a particle". Write "model the crate as a particle". When you're asked to criticise a model, name one specific way the assumption fails for *this* object. "It isn't realistic" earns nothing.

WORKED EXAMPLE 1.1 • STATING AND CRITICISING ASSUMPTIONS

A child on a sledge is pulled across snow by a rope. The sledge and child are modelled as a particle, the rope as light and inextensible, and the snow as smooth. Explain what each assumption means here, and say which is the least realistic.

Particle: the child and sledge are treated as a single point, so their size, the way the weight is spread, and any rotation or tipping are ignored.

Light, inextensible rope: the rope's weight is ignored, so the tension is the same all along it. It doesn't stretch, so the sledge moves exactly as the end of the rope does.

Smooth snow: there is no friction between the sledge and the snow.

The **least realistic** is "smooth". Snow does resist a sledge, and ignoring friction means the model will predict a larger acceleration than the real sledge has.

IN CONTEXT

Refine the model by treating the snow as rough and including a friction force.

3. Quantities and units

Every quantity in mechanics is a number *with a unit*. The **SI system** builds every unit from three base units.

KEY RESULT

Quantity	SI unit	Built from
length, displacement	metre, m	base unit
mass	kilogram, kg	base unit
time	second, s	base unit
speed, velocity	m s^{-1}	length $\div$ time
acceleration	m s^{-2}	velocity $\div$ time
force, weight	newton, N	$1 \text{ N} = 1 \text{ kg m s}^{-2}$ (mass $\times$ acceleration)

Exam papers write units as m s^{-1} rather than m/s . The negative index means “per”, exactly as in Chapter 1 of Pure: $\text{s}^{-1} = \frac{1}{\text{s}}$.

To **convert**, multiply by a fraction equal to 1, so the unwanted unit cancels.

WORKED EXAMPLE 1.2 • CONVERTING UNITS

(a) Convert 72 km h^{-1} to m s^{-1} . (b) The density of a material is 1.5 g cm^{-3} . Write this in kg m^{-3} .

(a) There are 1000 m in a km and 3600 s in an hour:

$$72 \text{ km h}^{-1} = \frac{72 \times 1000 \text{ m}}{3600 \text{ s}} = 20 \text{ m s}^{-1}$$

(b) $1 \text{ g} = 10^{-3} \text{ kg}$ and $1 \text{ cm}^3 = (10^{-2} \text{ m})^3 = 10^{-6} \text{ m}^3$:

$$1.5 \text{ g cm}^{-3} = \frac{1.5 \times 10^{-3} \text{ kg}}{10^{-6} \text{ m}^3} = 1500 \text{ kg m}^{-3}$$

QUICK CHECK: UNITS AS ALGEBRA

Units obey the index laws, so you can check a formula by checking its units. In $s = ut + \frac{1}{2}at^2$, the term ut has units $\text{m s}^{-1} \times \text{s} = \text{m}$, and at^2 has units $\text{m s}^{-2} \times \text{s}^2 = \text{m}$. Both match s , so the formula is at least consistent. A formula whose terms have different units must be wrong.

Mass and weight

Mass measures how much matter there is (kg). **Weight** is the force of gravity on that mass (N), and it acts vertically downwards.

KEY RESULT

$$W = mg, \quad g = 9.8 \text{ m s}^{-2}$$

Use $g = 9.8 \text{ m s}^{-2}$ unless a question says otherwise. An answer that uses g should be given to 2 or 3 significant figures, because g is only known to 2.

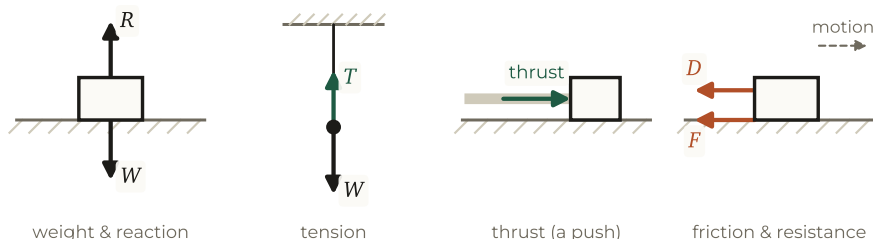
A 5 kg mass has weight $5 \times 9.8 = 49 \text{ N}$. A 750 g mass has weight $0.75 \times 9.8 = 7.35 \approx 7.4 \text{ N}$. Convert grams to kilograms *first*.

COMMON MISTAKE

“Weighs 5 kg” is fine in a shop but wrong in mechanics: 5 kg is a *mass*, and its *weight* is 49 N. And g is an acceleration, not a force. The force is mg .

4. The forces you'll meet

A **force diagram** shows every force acting *on* one object, drawn as arrows from that object. Each type of force has its own letter by convention, and knowing the types is most of the work of drawing one.



The forces of Year 1: weight and reaction, tension, thrust, and the resistances.

- **Weight** W (or mg): acts vertically downwards, from the centre of mass.
- **Normal reaction** R : a surface pushes back on an object resting on it. It acts at right angles to the surface.
- **Tension** T : a taut string or rope *pulls* on each object it's attached to.
- **Thrust** (compression): a rod *pushes* on the objects at its ends.
- **Friction** F : acts along a rough surface, opposing motion or any tendency to move.

- **Air resistance** D (or R): opposes motion through the air.

Chapter 3 of this course turns force diagrams into equations with Newton's laws. For now, the skill is to spot every force and to draw each one in the right direction.

5. Scalars, vectors and velocity

A **scalar** has size only. A **vector** has size *and* direction. Several mechanics words come in pairs, one of each:

Scalar (size only)	Vector (size and direction)
distance	displacement
speed	velocity
mass	weight (a force)
time	acceleration

A runner who completes one 400 m lap has run a *distance* of 400 m, but their *displacement* is zero: they finish where they started.

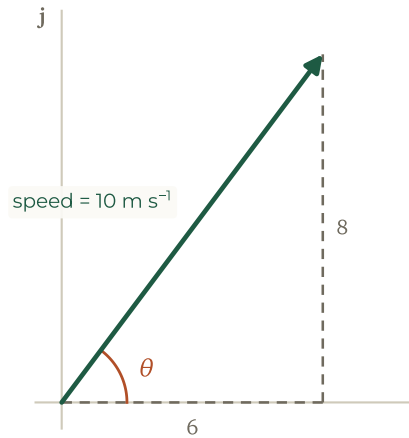
Vectors in mechanics use the $\mathbf{i}$, $\mathbf{j}$ notation from Chapter 15 of Pure. $\mathbf{i}$ usually points east (or right) and $\mathbf{j}$ north (or up).

KEY RESULT

For a velocity $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$:

$$\text{speed} = |\mathbf{v}| = \sqrt{a^2 + b^2}, \quad \tan \theta = \frac{b}{a}$$

where θ is the angle $\mathbf{v}$ makes with $\mathbf{i}$. A body moving with constant velocity $\mathbf{v}$ from position $\mathbf{r}_0$ is at $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ after time t .



Velocity $6\mathbf{i} + 8\mathbf{j} \text{ m s}^{-1}$: speed $\sqrt{6^2 + 8^2} = 10 \text{ m s}^{-1}$, at 53.1° to $\mathbf{i}$.

WORKED EXAMPLE 1.3 • SPEED AND DIRECTION

A particle moves with velocity $(5\mathbf{i} - 12\mathbf{j}) \text{ m s}^{-1}$. Find its speed, and the angle its direction of motion makes with $\mathbf{i}$.

The speed is the magnitude of the velocity:

$$|\mathbf{v}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13 \text{ m s}^{-1}$$

Sketch it: 5 right, 12 down. The angle below $\mathbf{i}$ satisfies

$$\tan \theta = \frac{12}{5} \Rightarrow \theta = 67.4^\circ$$

IN CONTEXT

The particle moves at 13 m s^{-1} , in a direction 67.4° below the $\mathbf{i}$ direction.

WORKED EXAMPLE 1.4 • BEARINGS AND POSITION

A boat leaves a harbour O with constant velocity $(3\mathbf{i} + 4\mathbf{j}) \text{ km h}^{-1}$, where $\mathbf{i}$ points east and $\mathbf{j}$ north. Find (a) its speed, (b) its bearing, and (c) its position vector after 4 hours if it started at $(1\mathbf{i} - 2\mathbf{j}) \text{ km}$ from O .

(a) $\sqrt{3^2 + 4^2} = 5 \text{ km h}^{-1}$.

(b) A bearing is measured clockwise from north, so use the angle between the velocity and $\mathbf{j}$. It is 3

east for every 4 north:

$$\tan \alpha = \frac{3}{4} \Rightarrow \alpha = 36.9^\circ, \text{ a bearing of } 036.9^\circ$$

$$\text{(c) } \mathbf{r} = \mathbf{r}_0 + \mathbf{v}t = (\mathbf{i} - 2\mathbf{j}) + 4(3\mathbf{i} + 4\mathbf{j}) = (13\mathbf{i} + 14\mathbf{j}) \text{ km.}$$

AVERAGE VELOCITY

If a body moves from A to B in time t , its **average velocity** is $\frac{\overrightarrow{AB}}{t}$ and its **average speed** is $\frac{\text{distance travelled}}{t}$. For a straight journey from $(2, 1)$ to $(14, 6)$ in 4 s, the displacement is $12\mathbf{i} + 5\mathbf{j}$.

That gives an average velocity of $(3\mathbf{i} + 1.25\mathbf{j}) \text{ m s}^{-1}$ and an average speed of $\frac{13}{4} = 3.25 \text{ m s}^{-1}$. On a curved or out-and-back journey, the two can be very different, as the 400 m lap shows.

Practice

Twelve questions in three tiers. Try each one properly before opening the solution. The solutions show the full method, not just answers. Take $g = 9.8 \text{ m s}^{-2}$ throughout.

Warm-up

Question 1

[3 marks]

Convert:

- (a) 54 km h^{-1} to m s^{-1}
- (b) 450 g to kg
- (c) 6 m s^{-1} to km h^{-1}

SOLUTION

$$\text{(a) } \frac{54 \times 1000}{3600} = 15 \text{ m s}^{-1}.$$

$$\text{(b) } 450 \div 1000 = 0.45 \text{ kg.}$$

$$\text{(c) In one hour the body goes } 6 \times 3600 = 21\,600 \text{ m} = 21.6 \text{ km, so the speed is } 21.6 \text{ km h}^{-1}.$$

Question 2

[3 marks]

- (a) Find the weight of a 3 kg mass.
- (b) Find the weight of a 250 g mass.
- (c) A box has weight 196 N. Find its mass.

SOLUTION

- (a) $W = mg = 3 \times 9.8 = 29.4 \text{ N}$.
- (b) Convert to kilograms first: $W = 0.25 \times 9.8 = 2.45 \text{ N}$.
- (c) $m = \frac{W}{g} = \frac{196}{9.8} = 20 \text{ kg}$.

Question 3

[3 marks]

A particle hangs at rest on the end of a string. The string is modelled as **light** and **inextensible**. Explain what each of these words means, and what each one allows you to ignore.

SOLUTION

Light: the string's mass is negligible. You can ignore its weight, so the tension is the same all the way along it.

Inextensible: the string does not stretch. You can ignore any change in its length, so anything attached to it moves exactly as its end does.

Question 4

[3 marks]

A particle moves with velocity $(-5\mathbf{i} + 12\mathbf{j}) \text{ m s}^{-1}$. Find its speed, and the angle between its direction of motion and the vector $\mathbf{i}$.

SOLUTION

$$\text{Speed} = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13 \text{ m s}^{-1}.$$

Sketch it: 5 left, 12 up. The acute angle with $-\mathbf{i}$ is $\tan^{-1} \frac{12}{5} = 67.4^\circ$, so the angle with $\mathbf{i}$ is $180^\circ - 67.4^\circ = 112.6^\circ$.

Standard**Question 5**

[4 marks]

An athlete runs exactly one lap of a 400 m track in 80 s, finishing where she started.

- (a) State the distance she ran and her displacement.
- (b) Find her average speed and her average velocity.
- (c) Use your answers to explain the difference between a scalar and a vector.

SOLUTION

(a) Distance = 400 m. Displacement = 0 m, because she ends where she began.

(b) Average speed = $\frac{400}{80} = 5 \text{ m s}^{-1}$. Average velocity = $\frac{\text{displacement}}{\text{time}} = \frac{0}{80} = 0 \text{ m s}^{-1}$.

(c) Distance and speed are scalars. They measure *how much*, whatever the direction, so every metre counts. Displacement and velocity are vectors. They account for direction, so running back to the start undoes the outward journey.

Question 6

[4 marks]

A particle P starts at the point with position vector $(-2\mathbf{i} + 5\mathbf{j}) \text{ m}$ and moves with constant velocity $(3\mathbf{i} - 2\mathbf{j}) \text{ m s}^{-1}$.

- (a) Find the position vector of P after 6 seconds.
- (b) Find the distance of P from the origin at that time.

SOLUTION

(a) $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t = (-2\mathbf{i} + 5\mathbf{j}) + 6(3\mathbf{i} - 2\mathbf{j}) = (16\mathbf{i} - 7\mathbf{j}) \text{ m}$.

(b) $|\mathbf{r}| = \sqrt{16^2 + (-7)^2} = \sqrt{305} = 17.5 \text{ m (3 s.f.)}$.

Question 7

[5 marks]

A crate is dragged across a warehouse floor by a rope. In a first model, the crate is a particle, the rope is light, and the floor is smooth.

- (a) Draw a force diagram for the crate in this model, naming each force.
- (b) State which assumption is least realistic, and the effect it has on the acceleration the model predicts.
- (c) Suggest how the model could be refined.

SOLUTION

- (a) Three forces act at a single point: the weight W vertically down, the normal reaction R vertically up, and the tension T along the rope. There is no friction arrow, because the floor is smooth.
- (b) “Smooth” is least realistic. A crate on a warehouse floor meets significant friction. With the resistance to motion left out, the model predicts a **larger** acceleration than the crate really has.
- (c) Model the floor as rough, and include a friction force F opposing the motion.

IN CONTEXT

With friction included, the model’s predicted acceleration drops towards what is actually observed.

Question 8

[4 marks]

Using $1 \text{ N} = 1 \text{ kg m s}^{-2}$:

- (a) Momentum is mass $\times$ velocity. Find its SI units.
- (b) Work done is force $\times$ distance. Write its units in terms of kg, m and s.
- (c) Show that every term in $v^2 = u^2 + 2as$ has the same units, where u and v are velocities, a an acceleration and s a displacement.

SOLUTION

- (a) $\text{kg} \times \text{m s}^{-1} = \text{kg m s}^{-1}$.
- (b) $\text{N} \times \text{m} = \text{kg m s}^{-2} \times \text{m} = \text{kg m}^2 \text{ s}^{-2}$.
- (c) v^2 and u^2 have units $(\text{m s}^{-1})^2 = \text{m}^2 \text{ s}^{-2}$. The term $2as$ has units $\text{m s}^{-2} \times \text{m} = \text{m}^2 \text{ s}^{-2}$ (the 2 is a pure number). All three terms match. ■

Exam style**Question 9**

[6 marks]

In this question $\mathbf{i}$ points due east and $\mathbf{j}$ due north. A boat leaves a harbour O and moves with constant velocity $(4\mathbf{i} + 3\mathbf{j}) \text{ km h}^{-1}$.

- (a) Find the speed of the boat.
- (b) Find the bearing on which the boat moves.
- (c) Find how far the boat is from O after 2 hours 30 minutes.

SOLUTION

(a) $\sqrt{4^2 + 3^2} = 5 \text{ km h}^{-1}$.

(b) The bearing is measured clockwise from north. The boat goes 4 east for every 3 north, so the angle α from north satisfies $\tan \alpha = \frac{4}{3}$, giving $\alpha = 53.1^\circ$. The bearing is 053° (to the nearest degree).

(c) After 2.5 hours, $\mathbf{r} = 2.5(4\mathbf{i} + 3\mathbf{j}) = (10\mathbf{i} + 7.5\mathbf{j}) \text{ km}$, so the distance is $\sqrt{10^2 + 7.5^2} = 12.5 \text{ km}$.

(Or: the speed is constant at 5 km h^{-1} along a straight line, so the distance is $5 \times 2.5 = 12.5 \text{ km}$.)

Question 10

[6 marks]

At time $t = 0$, particle A is at $(2\mathbf{i} + \mathbf{j}) \text{ m}$ and moves with constant velocity $(3\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-1}$. At the same moment, particle B is at $(10\mathbf{i} - 7\mathbf{j}) \text{ m}$ and moves with constant velocity $(\mathbf{i} + 4\mathbf{j}) \text{ m s}^{-1}$.

- (a) Write the position vectors of A and B at time t seconds.
- (b) Show that A and B collide, and find the time and the position of the collision.

SOLUTION

(a) $\mathbf{r}_A = (2 + 3t)\mathbf{i} + (1 + 2t)\mathbf{j}$ and $\mathbf{r}_B = (10 + t)\mathbf{i} + (-7 + 4t)\mathbf{j}$.

(b) They collide if both components are equal at the *same* time.

i: $2 + 3t = 10 + t \Rightarrow t = 4$.

j: $1 + 2t = -7 + 4t \Rightarrow t = 4$.

Both components give $t = 4$, so the particles are at the same point at the same time.

IN CONTEXT

They collide after 4 seconds, at the point with position vector $(14\mathbf{i} + 9\mathbf{j}) \text{ m}$.

Question 11

[6 marks]

A stone is dropped from rest from the top of a cliff 45 m high. A student models the stone as a particle falling freely under gravity, which gives a time to reach the sea of $t = \sqrt{\frac{2h}{g}}$, where h is the height fallen.

- (a) Find the time the model predicts.
- (b) The stone actually takes 3.4 s. Comment on the model, naming the assumption most likely to be responsible.
- (c) Another student uses $g = 10 \text{ m s}^{-2}$. Find their prediction, and explain why changing g is not a

sensible way to improve the model.

SOLUTION

$$(a) t = \sqrt{\frac{2 \times 45}{9.8}} = \sqrt{9.18\dots} = 3.03 \text{ s (3 s.f.)}.$$

(b) The model predicts a shorter time than was observed, so it underestimates the time by about 0.4 s. The most likely cause is ignoring **air resistance**: treating the stone as a particle falling freely leaves out the drag that slows it down. A refined model would include a resistance force.

(c) $t = \sqrt{\frac{90}{10}} = 3 \text{ s}$. This moves the prediction the wrong way, further from 3.4 s. More importantly, g is a measured physical constant, not a number to tune. The mismatch comes from a missing force, so that is what the refinement should add.

Question 12

[7 marks]

A particle P moves with constant velocity. At $t = 0$ it is at the point A with position vector $(-3\mathbf{i} + 4\mathbf{j})$ m. At $t = 5 \text{ s}$ it is at B , with position vector $(12\mathbf{i} - 4\mathbf{j})$ m.

- Find the velocity of P .
- Find the speed of P .
- Find the angle between the direction of motion and the vector $\mathbf{i}$.
- Find the time at which P is due east of the origin (i.e. its $\mathbf{j}$ -component is zero), and its position then.

SOLUTION

$$(a) \mathbf{v} = \frac{\overrightarrow{AB}}{5} = \frac{(12\mathbf{i} - 4\mathbf{j}) - (-3\mathbf{i} + 4\mathbf{j})}{5} = \frac{15\mathbf{i} - 8\mathbf{j}}{5} = (3\mathbf{i} - 1.6\mathbf{j}) \text{ m s}^{-1}.$$

$$(b) \sqrt{3^2 + 1.6^2} = \sqrt{11.56} = 3.4 \text{ m s}^{-1}.$$

(c) The velocity points 3 right and 1.6 down, so the angle below $\mathbf{i}$ is $\tan^{-1} \frac{1.6}{3} = 28.1^\circ$.

(d) $\mathbf{r} = (-3 + 3t)\mathbf{i} + (4 - 1.6t)\mathbf{j}$. Set the $\mathbf{j}$ -component to zero: $4 - 1.6t = 0 \Rightarrow t = 2.5 \text{ s}$. Then the $\mathbf{i}$ -component is $-3 + 3(2.5) = 4.5$.

IN CONTEXT

P is due east of the origin after 2.5 s, at $4.5\mathbf{i} \text{ m}$, i.e. 4.5 m east of O .